Darboux transformations in theory of integrable systems

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What is integrability?

Different definitions of integrability are used for different types of equations. There are two most general approaches: to consider properties of the equation itself, or properties of its solutions.

Properties of equations:

- explicit integrability
- sufficient number of first integrals in involution (ODEs)
- existence of a Lax representation
- sufficient number of integrals along characteristics (hyperbolic PDEs in 2D)
- existence of higher symmetries
- 3D-consistency (PΔEs on quad-graphs in 2D)
- ...

First integrals

Function $F(x_1, x_2, \dots, x_n)$ is called a *first integral* of the dynamical system

$$\begin{cases} \dot{x}^1 = f_1(x^1, x^2, \dots, x^n) \\ \dot{x}^2 = f_2(x^1, x^2, \dots, x^n) \\ \dots \\ \dot{x}^n = f_n(x^1, x^2, \dots, x^n) \end{cases}$$

if it is constant along the trajectories: $\frac{d}{dt}(F) = 0$.

Existence of a first integral allows to reduce the order of the system by 1.

Important remark

According to standard rectification theorem in a neighborhood of any non-singular point of a smooth vector field there is a coordinate system such that this vector field becomes constant.

But that's not the integrability we are going to discuss!

Hamiltonian systems

A skew-symmetric tensor field ω^{ij} defines a *Poisson bracket*

$$\{f, g\} = \sum_{i,j=1}^{n} \omega^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}$$

on a manifold M if the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

holds for arbitrary smooth functions $f, g, h \in C^\infty(M)$. A dynamical system

$$\dot{x}^1 = \{H, x^1\},\ \dot{x}^2 = \{H, x^2\}, \dots, \dot{x}^n = \{H, x^n\}$$

is called *hamiltonian*.

Liouville's Theorem

Consider a hamiltonian system on a symplectic manifold M, $2n = \dim M$, and let $I_1, I_2, \ldots, I_n$ be it's first integrals such that each pair of them is *in involution*:

$$\{I_i, I_j\} = 0.$$

If a common level set

$$M_{\mathbf{c}} = \{I_1 = c_1, I_2 = c_2, \ldots, I_n = c_n\}$$

is compact, connected and all integrals $I_1, I_2, \ldots, I_n$ are functionally independent on it, then $M_{\mathbf{c}}$ is diffeomorphic to n-dimensional torus and hamiltonian system can be integrated in quadratures.

Example: finite Toda chain

The *Toda chain*

$$\begin{cases} \ddot{q}_1 = -e^{q_1 - q_2} \\ \ddot{q}_j = e^{q_{j-1} - q_j} - e^{q_j - q_{j+1}}, \quad j = 2, 3, \ldots, n-1 \\ \ddot{q}_n = e^{q_{n-1} - q_n} \end{cases}$$

is hamiltionian with respect to the canonical Poisson bracket and the Hamiltomian

$$H = \frac{1}{2} \sum_{j=1}^{n} p_j^2 + \sum_{j=1}^{n-1} e^{q_j - q_{j+1}},$$

where $p_j = \dot{q}_j$.

Toda chain, Lax representation

The Toda chain admits a *Lax representation*, i.e. it is equivalent to matrix equation:

$$L_t = [A, L],$$

where

$$L = \begin{pmatrix} a_1 & b_1 & \dots & 0 \\ b_1 & a_2 & \dots & 0 \\ \vdots & \ddots & \ddots & b_{n-1} \\ 0 & \dots & b_{n-1} & a_n \end{pmatrix}, \; A = \begin{pmatrix} 0 & b_1 & \dots & 0 \\ -b_1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & b_{n-1} \\ 0 & \dots & -b_{n-1} & 0 \end{pmatrix},$$

$$a_j = -\frac{1}{2}p_j, \quad j = 1, 2, \dots, n, \quad b_j = \frac{1}{2}e^{\frac{1}{2}(q_j - q_{j+1})}, \quad j = 1, 2, \dots, n-1.$$

Toda chain: Liouville integrability

Theorem

Lax representation provides the sufficient number of first integrals in involution:

$$I_k = \operatorname{tr} L^k, \quad k = 1, 2, \ldots, n.$$

The fact that I_1 is a first integrals is obvious:

$$\frac{d}{dt}(I_1) = \frac{d}{dt}(\operatorname{tr} L) = \operatorname{tr} L_t = \operatorname{tr}[A, L] = 0.$$

If $k > 1$, then this follows from the fact that all eigenfunctions of the Lax matrix L are first integrals.

Darboux transformations

Linear differential operators

$$L = \frac{d^n}{dx^n} + a_{n-1}\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1\frac{d}{dx} + a_0$$

and

$$\hat{L} = \frac{d^n}{dx^n} + \hat{a}_{n-1}\frac{d^{n-1}}{dx^{n-1}} + \cdots + \hat{a}_1\frac{d}{dx} + \hat{a}_0$$

are related by a *Darboux transformation* if there exists a differential operator D such that

$$\hat{L} \circ D = D \circ L.$$

The main property of Darboux transformations is that D maps $\operatorname{Ker} L$ to $\operatorname{Ker} \hat{L}$.

Decomposition theorem

Theorem

Any Darboux transformation of the order r can be represented as a composition of first-order Darboux transformations, i.e. if

$$\hat{L} \circ D = D \circ L,$$

then there exist a sequence of operators L_j, such that $L_0 = L$, $L_r = \hat{L}$ and

$$L_{j+1} \circ D_j = D_j \circ L_j, \quad j = 0, 1, \ldots, r-1,$$

where $D_j = \frac{d}{dx} + \gamma_j$. In this case, $D = D_r D_{r-1} \ldots D_1$.

Veselov-Shabat dressing chain

Consider a Shrödinger operator $L_0 = -\frac{d^2}{dx^2} + u_0(x)$, it can be factorized as follows:

$$L_0 = D_0 D_0^+ - \alpha_0, \ \alpha_0 = \mathrm{const},$$

where $D_0 = d/dx + f_0(x)$ and $D_0^+ = -d/dx + f_0(x)$. Define a new operator and refactorize it in the same way:

$$L_1 = D_0^+ D_0 = -\frac{d^2}{dx^2} + u_1 = D_1 D_1^+,$$

etc. On each step operators L_j and $L_{j+1} - \alpha_j$ are related by a Darboux transformation:

$$(L_{j+1} - \alpha_j) D_j^+ = D_j^+ L_j.$$

Veselov-Shabat dressing chain, continued

Periodic closure $L_r = L_0$ of this sequence is equivalent to the following system of ODEs:

$$\begin{cases} (f_r + f_1)' = f_r^2 - f_1^2 + \alpha_1 \\ (f_1 + f_2)' = f_1^2 - f_2^2 + \alpha_2 \\ \dots \\ (f_{r-1} + f_r)' = f_{r-1}^2 - f_r^2 + \alpha_r \end{cases} .$$

This system is called the *Veselov-Shabat dressing chain*. It is related to the famous KdV equation.

Denote $\alpha = \sum_{j=1}^{r} \alpha_j$. Cases $\alpha = 0$ and $\alpha \neq 0$ are essentially different.

Dressing chain: Lax representation

Suppose $\alpha = 0$. Consider an eigenfunction ψ_0 such that $L_0\psi_0 = \lambda\psi_0$ for some $\lambda \in \mathbb{R}$ and denote $\psi_j = D^+_{j-1}\psi_j$, $j = 1, 2, \ldots, r$. Hence $L_j\psi = \lambda_j\psi_j$, where

$$\lambda_j = \lambda - \alpha_1 - \alpha_2 - \cdots - \alpha_j, \quad j = 1, 2, \ldots, r.$$

The scalar equation $-\psi_j'' + u_j\psi_j = \lambda_j\psi_j$ is equivalent to the matrix equation $\Psi_j' = U_j\Psi_j$, where $\Psi_j = (\psi_j, \psi_j')^T$,

$$U_j = \begin{pmatrix} 0 & 1 \\ u_j - \lambda_j & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ f_j' + f_j^2 - \lambda + \beta_j & 0 \end{pmatrix}.$$

and $\alpha_j = \beta_{j-1} - \beta_j$.

Dressing chain: Lax representation, continued

The equation $\psi_{j+1} = D_j^+ \psi_j$ is equivalent to the matrix equation $\Psi_{j+1} = W_j \Psi_j$, where

$$W_j = \begin{pmatrix} f_j & -1 \\ f_j' - u_j + \lambda_j & f_j \end{pmatrix} = \begin{pmatrix} f_j & -1 \\ f_j' - u_j + \lambda - \beta_j & f_j \end{pmatrix}.$$

On one hand,

$$\Psi_{j+1}' = (W_j \Psi_j)' = W_j' \Psi_j + W_j \Psi_j' = W_j' \Psi_j + W_j U_j \Psi_j = \left(W_j' + W_j U_j\right) \Psi_j,$$

and on the other hand,

$$\Psi_{j+1}' = U_{j+1} \Psi_{j+1} = U_{j+1} W_j \Psi_j.$$

Hence, the dressing chain is equivalent to the sequence of matrix equations

$$W_j' = U_{j+1} W_j - W_j U_j.$$

Dressing chain: Lax representation, continued

Theorem

[Shabat, Yamilov, 1990] *If* $\alpha = 0$*, then the periodic dressing chain admits a parametric dependent Lax representation, i.e. it is equivalent to the following matrix equation*

$$\frac{d}{dx} W(\lambda) = [U_1, W(\lambda)],$$

where $W(\lambda) = W_n W_{r-1} \dots W_1$*.*

Due to the obvious fact that $\operatorname{tr} W'(\lambda) = 0$, such Lax representation provides a generating function for first integrals:

$$\tau(\lambda) = \operatorname{tr} W(\lambda) = (-1)^r (I_0 + I_1 \lambda + I_2 \lambda^2 + \dots + I_r).$$

Note that these integrals are a direct consequence of the structure of the dressing chain related to Darboux transformations.

Dressing chain: Liouville integrability

Theorem

[Veselov, Shabat, 1993] *Dressing chain is a hamiltonian system with respect to the Poisson structure*

$$\omega^{ij} = (-1)^{i-j (mod\, r)} \text{ if } j \neq i, \quad \omega^{ii} = 0$$

and with the following hamiltonian:

$$H = \sum_{j=1}^{r} \left(\frac{1}{3} f_j^3 + \beta_j f_j \right).$$

If $\alpha = 0$ *and* r *is odd, then it is Liouville integrable.*

Integrability follows from the fact that the first integrals provided by the Lax representation appear to be in involution.

Laplace invariants

Definition

Functions $h = b_y - ab - c$ and $k = a_x - ab - c$ are called *the Laplace invariants* of the hyperbolic differential operator

$$L = \partial_x \partial_y + a\partial_x + b\partial_y + c.$$

Hyperbolic operator can be factorized if one the Laplace invariants is zero:

$$L = (\partial_x + b)(\partial_y + a) + k = (\partial_y + a)(\partial_x + b) + h.$$

2D-Toda lattice

Consider a sequence of hyperbolic operators

$$L_j = \partial_x \partial_y + a(j)\partial_x + b(j)\partial_y + c(j)$$

such that any two neighboring operators are related by a *Darboux-Laplace transformation*:

$$L_{j+1} D_j = D_{j+1} L_j,$$

where $D_j = \partial_x + b(j)$. Then the Laplace invariants of these operators satisfy the *two-dimensional Toda lattice*:

$$(\ln h(j))_{xy} = h(j+1) - 2h(j) + h(j-1).$$

Different forms of 2D-Toda lattice

Toda lattice can be represented in various forms in terms of different sets of variables.

- $q_{xy}(j) = \exp(q(j+1) - q(j)) - \exp(q(j) - q(j-1))$,
- or
$$(\ln h(j))_{xy} = h(j+1) - 2h(j) + h(j-1),$$
where $h(j) = \exp(q(j+1) - q(j))$,
- or
$$u_{xy}(j) = \exp(u(j+1) - 2u(j) + u(j-1))$$
where $h(j) = u_{xy}(j)$.

How to obtain a finite system?

- Trivial boundary conditions: $u(-1) = u(r) = -\infty$.
- Periodic boundary conditions: $u(j + r) = u(j)$.

These Toda lattices are particular cases of the so-called *exponential system*:

$$u_{xy}(i) = \exp\left(\sum_{j=1}^{r} a_{ij} u(j)\right), \quad i = 1, 2, \ldots, r,$$

where $a_{ij} = \mathrm{const}$.
Shabat, Yamilov (1981): What exponential systems are integrable?

Integrable boundary conditions

- Trivial boundary conditions correspond to the matrix

$$M = (a_{ij}) = \begin{pmatrix} -2 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & \dots & 0 \\ 0 & 1 & -2 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -2 \end{pmatrix}.$$

- Notice that $-M$ is the Cartan matrix of an A-series Lie algebra. There are other simple Lie algebras!
- $\approx$ 1980: Many papers by various authors on finite 2D-Toda lattices.

Integrals along characteristics

Definition

Function $I = I(\mathbf{u}_x, \mathbf{u}_{xx}, \mathbf{u}_{xxx}, \dots)$ is called *y-integral* of the system

$$\mathbf{u}_{xy} = F(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_y)$$

if $D_y(I) = 0$ on solutions of the system. Integrals in direction x are defined similarly.

Example

Functions $I = u_{xx} - \frac{1}{2}u_x^2$ and $J = u_{yy} - \frac{1}{2}u_y^2$ are y- and x-integrals resp. of the Liouville equation $u_{xy} = e^u$.

Darboux integrability

Definition

Integrals $I_1, I_2, \ldots, I_k$ of orders $d_1, d_2, \ldots, d_k$ resp. are called *essentially independent*, if

$$\mathrm{rk}\left(\frac{\partial I_i}{\partial(\partial_x^{d_i} u_j)}\right) = k.$$

Hyperbolic system

$$\mathbf{u}_{xy} = F(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_y)$$

is called *Darboux integrable* if it admits complete families of essentially independent x- and y-integrals.

Generating function for y-integrals

Theorem

[S., 2015] Coefficients of the differential operator

$$\mathcal{B} = (\partial_x - q_x(r))(\partial_x - q_x(r-1))\dots(\partial_x - q_x(0))$$

are y-integrals of the A-series Toda lattice defined by boundary conditions $q(-1) = +\infty$, $q(r) = -\infty$. These integrals are essentially independent.

Proof: use of DLTs

$$q(-1) = +\infty,\ q(r) = -\infty \Longrightarrow h(-1) = k(0) = 0,\ h(r) = 0$$

Therefore, $L_0 = (\partial_x + b(0))\partial_y$, $L_r = \partial_y(\partial_x + b(r))$. In the Toda lattice $b(j) = -q_x(j)$. Hence, $B = D_r D_{r-1} \dots D_0$.
Now use DLTs:

$$B \circ \partial_y = D_r D_{r-1} \dots D_0 \partial_y = D_r \dots D_1 L_0 = \\ D_r \dots L_1 D_0 = \dots = L_r D_{r-1} D_{r-2} \dots D_0 = \partial_y \circ B.$$

$[\partial_y, B] = 0 \Longrightarrow$ coeffs. of B do not depend on y.

□

Integrability of finite lattices

A-series Toda lattice equations are symmetric about the change of variables $x \leftrightarrow y$. Therefore, the similar construction allows to obtain x-integrals

$$J_1, J_2, \dots J_r$$

as well. These integrals are independent.
Explicit formulas that define reductions of an A-series lattice to B- or C-series lattice allows to prove the following

Theorem

[S., 2015] B- and C-series Toda lattices are Darboux integrable.

Discrete dressing chain

Consider second-order difference operators $L_j = A_j A_j^+ - \alpha_j$, related by Darboux transformations

$$(L_{j+1} - \alpha_j) A_j^+ = A_j^+ L_j,$$

where $A_j = a_j(n) + b_j(n) T$, $A_j^+ = a_j(n) + b_j(n-1) T^{-1}$ and T is a shift operator on 1D-lattice: $T\psi(n) = \psi(n+1)$. Periodic closure of this Darboux chain is equivalent to the system of difference equations:

$$\begin{cases} a_j^2(n) + b_j^2(n) = a_{j-1}^2(n) + b_{j-1}^2(n-1) + \alpha_j \\ a_j(n) b_j(n-1) = a_{j-1}(n-1) b_{j-1}(n-1) \end{cases} .$$

Darboux transformations imply a Lax representation for this system and therefore allow to obtain conserved quantities for the discrete dynamics [S., 2005].

Darboux q-chain

In the discrete case consider a modified operator relation

$$L_j = A_j A_j^+ - \alpha_j = q A_{j-1}^+ A_{j-1},$$

where $0 < q < 1$, with the following closure condition:

$$L_{j+r} = T^{-r} L_j T^r.$$

This system is called the *Darboux q-chain*. All operators L_j are bounded and have purely discrete spectrum that can be obtained using the Darboux scheme [Dynnikov, S., 2002, S.,2003]

Discrete Toda lattices

In the semidiscrete case DLTs for hyperbolic operators

$$L = \partial_x T + a\partial_x + bT + c,$$

lead to the following differential-difference equations

$$\left(\ln \frac{h_n(j)}{h_{n+1}(j)}\right)'_x = h_{n+1}(j+1) - h_{n+1}(j) - h_n(j) + h_n(j-1), \;\; n \in \mathbb{Z}.$$

Discrete Toda lattices, continued

In the purely discrete case

$$L_j = T_1 T_2 + a(j) T_1 + b(j) T_2 + c(j),$$

we obtain similarly the following system of PΔEs:

$$\frac{h_{n,m+1}(j) h_{n-1,m}(j)}{h_{n,m}(j) h_{n-1,m+1}(j)} = \frac{(1 + h_{n,m}(j+1))(1 + h_{n-1,m+1}(j-1))}{(1 + h_{n,m}(j))(1 + h_{n-1,m+1}(j))}.$$

In both case the DT approach allows to obtain a generating function for integrals along characteristics and therefore to prove Darboux integrability on the A-series lattice and of some its reductions [S., 2012, 2015].

Discretization using DTs

Consequent application of Darboux transformations to solutions of integrable models always produce a discrete system. In some cases it is trivial, but sometimes it is very nontrivial and inherits some properties of initial integrable system (Adler, Mikhailov, Sokolov,. . .)

Thank you!