Another ham sandwich in the plane

joint work with Alexey Balitskiy and Roman Karasev from Moscow Institute of Physics and Technology

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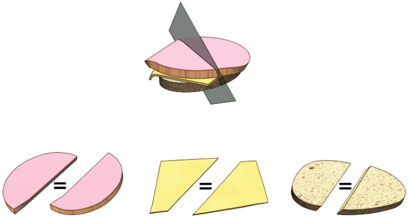
Ham sandwich theorem

Theorem (coffee-break version)

Every sandwich with ham and cheese can be cut by one planar cut in such way, that both pieces will contain equal amount of bread, ham, and cheese.



Ham sandwich theorem

Theorem (mathematical version, due to Stone and Tukey (1942) and Steinhaus (1945))

Every d "nice" measures in $\mathbb{R}^d$ can be cut by one hyperplane in such way, that each of two halfspaces will contain half of each measure.

Ham sandwich theorem

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Nice measure μ:

- measure of the whole space is 1 (or at least finite);
- measure of each hyperplane is 0. This property could be omitted but with additional restrictions.

Topological ancestors

Theorem (Borsuk-Ulam theorem)

For every continuous map $f : \mathbb{S}^d \longrightarrow \mathbb{R}^d$ *there exists a point* $\mathbf{x} \in \mathbb{S}^d$ *such that* $f(\mathbf{x}) = f(-\mathbf{x})$.

Theorem (Lyusternik-Shnirelman theorem)

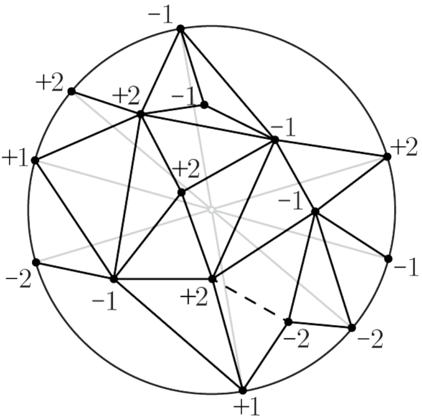
If the sphere $\mathbb{S}^d$ *is covered by* $d+1$ *closed sets, then at least one of these sets contains a pair of antipodal points.*

Topological ancestors

Theorem (Tucker's lemma)

Assume $\mathcal{T}$ is a triangulation of d-dimensional ball with all vertices labeled by numbers from the set $\{+1, -1, \dots, +d, -d\}$. If on the boundary this labeling is antipodal (i.e. set of vertices on the boundary is antipodal and a opposite points are labeled with opposite numbers) then there is an edge with vertices labeled with opposite numbers.

Topological ancestors



Equipartitions with fans

Definition

A *k-fan* is a collection of k rays on the plane with common initial point.

Theorem (Bárány, Matoušek, 2001)

Any three measures can be

- *simultaneously halved by a 2-fan;*
- *divided into parts* $(\frac{2}{3}, \frac{1}{3})$ *by a 2-fan.*

Equipartitions with fans

Theorem (Bárány, Matoušek, 2001)

Any two measures can be

- *divided into parts α and $1-\alpha$ by a 2-fan;*
- *equipartitioned by a 3-fan;*
- *divided into parts $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ by a 3-fan;*
- *divided into parts $(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$ by a 4-fan.*

Theorem (Bárány, Matoušek, 2002)

Any two measures can be divided into four equal parts each by a 4-fan.

Equipartitions into convex parts

Conjecture (Kaneko, Kano, 2002)

For any d measures in $\mathbb{R}^d$ and for any k the is a partition of $\mathbb{R}^d$ into k convex part such that each part contains $\frac{1}{k}$ of each measure.

This conjecture was independently proved by Soberón, 2010, and Karasev, 2010

Theorem (Akopyan, Karasev, 2013)

Under some additional restrictions for any $d+1$ measures in $\mathbb{R}^d$ one can

- *cut the same fraction from all measures by a halfspace;*
- *cut the same prescribed fraction from all measures by a convex set.*

Equipartitions into convex parts

Conjecture (Nandakumar, Ramano Rao, 2008)

For any n a convex body in $\mathbb{R}^d$ can be cut into n convex parts of equal d-dimensional volume and equal $(d-1)$-dimensional surface volume.

Aronov, Hubard, and Karasev in 2013 proved this conjecture for n equal to power of prime.

Equipartitions with polynomial surfaces

Theorem (Gromov, 2003)

If $n \geq 1$ then every $\binom{n+d}{n} - 1$ measures in $\mathbb{R}^d$ can be divided into halves by a polynomial surface of degree at most n.

Polyhedral curtain theorem by Rade Živaljević

Consider a d-simplex $\Delta = \mathrm{conv}\{\mathbf{x}_0, \ldots, \mathbf{x}_d\}$ contains the origin O in its interior.

Definition

A *polyhedral curtain* of Δ is the cone with vertex at O over join $\partial F_1 * \partial F_2$ of boundaries of two complementary faces of Δ. It is a cone over $(d-2)$-dimensional polyhedral sphere.

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How to construct curtains: each $(d-2)$-face of base contains all vertices of F_1 except one and all vertices of F_2 except one.

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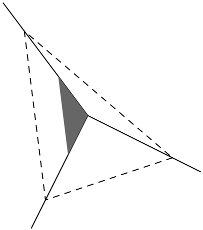
Theorem (Živaljević, 2013)

For any d measures in $\mathbb{R}^d$ there exist a **translation** *of some polyhedral curtain of Δ that divides all measures into equal parts.*

Difference in two- and three-dimensional cases

$\mathbb{R}^2$

Any curtain is an angle from face-fan of the simplex (planar 3-fan).



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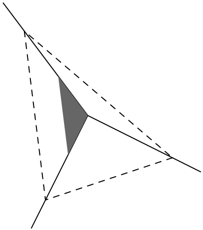
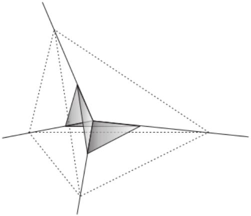


$\mathbb{R}^d$ with $d \geq 3$

Some curtains are constructed from several cones from face-fan of the simplex



Planar generalizations of polyhedral curtain theorem

Theorem (Balitskiy, A.G., Karasev, 2013)

Any two measures on the plane can be equipartitioned by a translation of some angle from an arbitrary k-fan for **odd** k.

Planar generalizations of polyhedral curtain theorem

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Any two measures on the plane can be equipartitioned by a translation of some angle from an arbitrary k-fan for **odd** k.

Theorem

Any two measures on the plane can be equipartitioned by a translation of some angle from an arbitrary **symmetric** $4k$*-fan.*

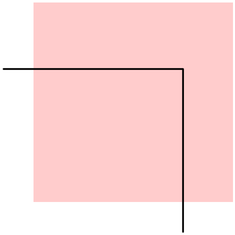
Theorem

Any two measures on the plane can be equipartitioned by a translation of some angle from an arbitrary k-fan if this fan contains **two opposite rays with even number of angles between them**.

Idea of proof

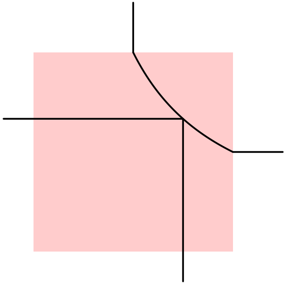
Consider a red measure

Idea of proof



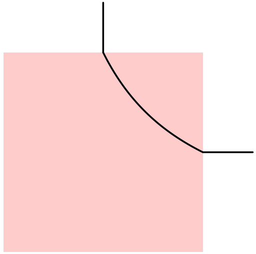
and an angle.

Idea of proof



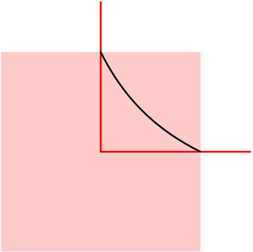
Draw the set of vertices of translated angle that cut the half of measure.

Idea of proof



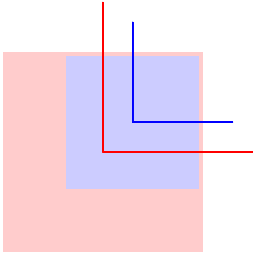
There are two “rays” that we can see in this set.

Idea of proof



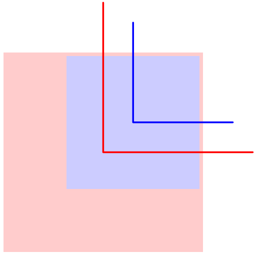
These rays defines the red angle corresponding to the initial red measure.

Idea of proof



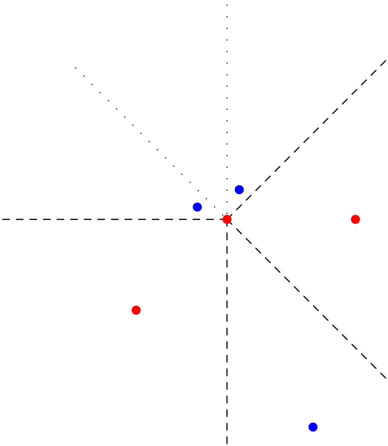
Repeat the same construction for the blue measure.

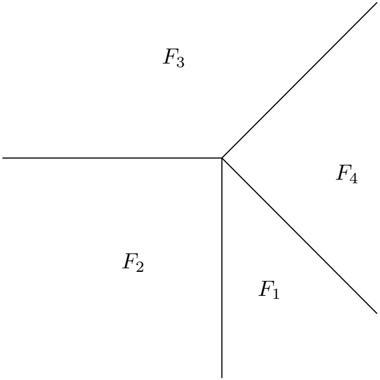
Idea of proof



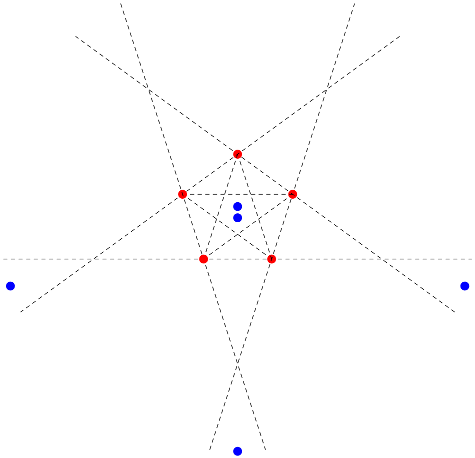
If there is no desired translation of initial angle then one of constructed angles contains another.

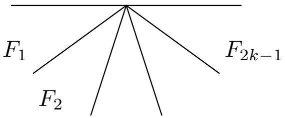
Some counterexamples: almost arbitrary 4-fan



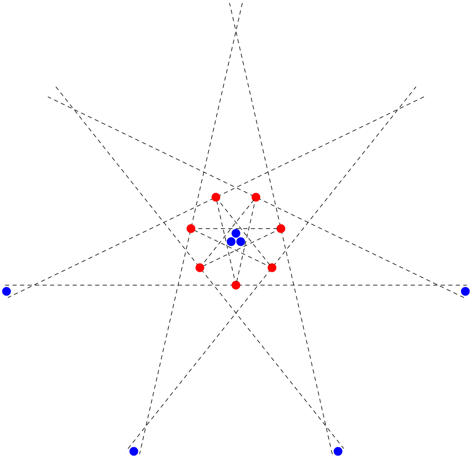


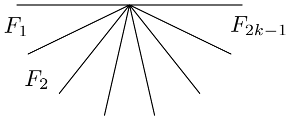
Some counterexamples: $(4k+2)$-fan





Some counterexamples: $4k$-fan





THANK YOU!