Parallelohedra and the Voronoi Conjecture

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Parallelohedra

Definition

Convex d-dimensional polytope P is called a **parallelohedron** if $\mathbb{R}^d$ can be (face-to-face) tiled into parallel copies of P.

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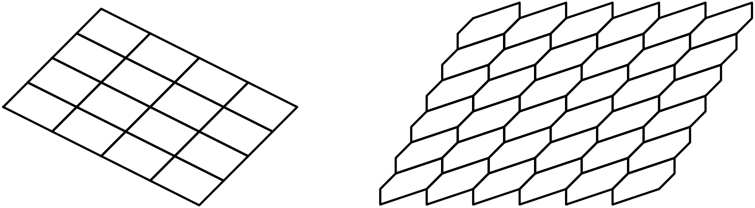

Two types of two-dimensional parallelohedra

Three-dimensional parallelohedra

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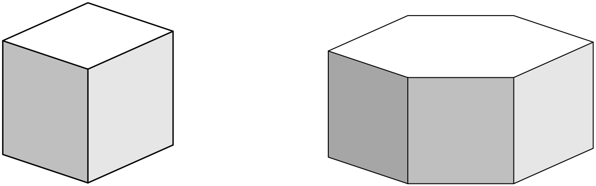

Parallelepiped and hexagonal prism with centrally symmetric base.

Three-dimensional parallelohedra

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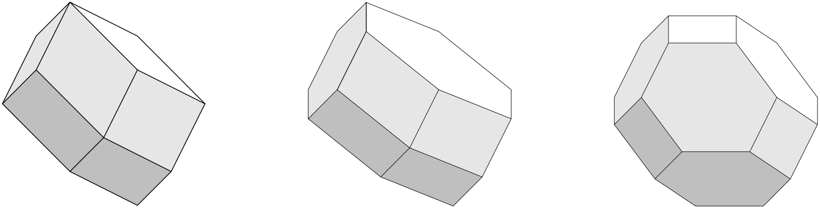
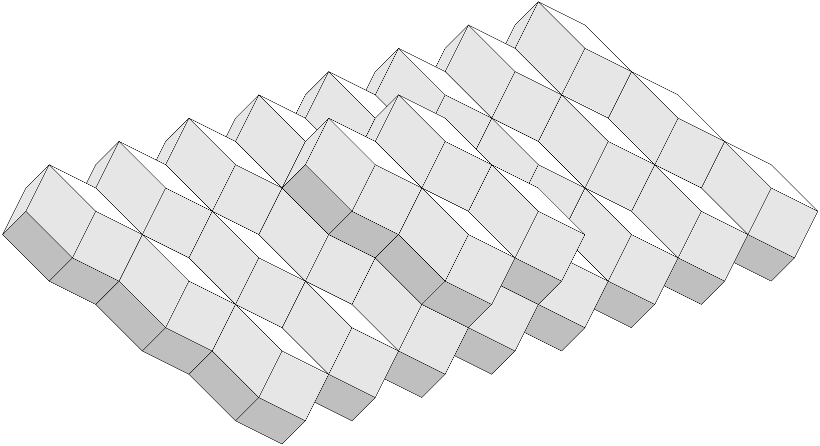


Rhombic dodecahedron, elongated dodecahedron, and truncated octahedron

Tiling by rhombic dodecahedra



Minkowski-Venkov conditions

Theorem (H.Minkowski, 1897, and B.Venkov, 1954)

P is a d-dimensional parallelohedron iff it satisfies the following conditions:

1. *P is centrally symmetric;*
2. *Any facet of P is centrally symmetric;*
3. *Projection of P along any its $(d-2)$-dimensional face is parallelogram or centrally symmetric hexagon.*

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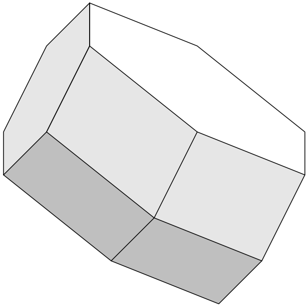
Theorem (N.Dolbilin and A.Magazinov, 2013)

If P tiles d-dimensional space with positive homothetic copies separated from 0 then P satisfies Minkowski-Venkov conditions.

Belts of parallelohedra

Definition

The set of facets parallel to a given $(d-2)$-face is called **belt**. These facets are projected onto sides of a parallelogram or a hexagon.



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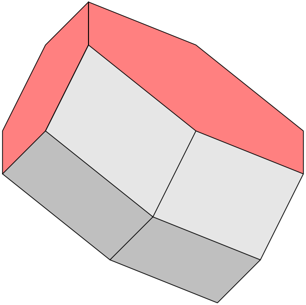

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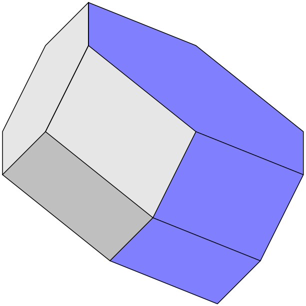
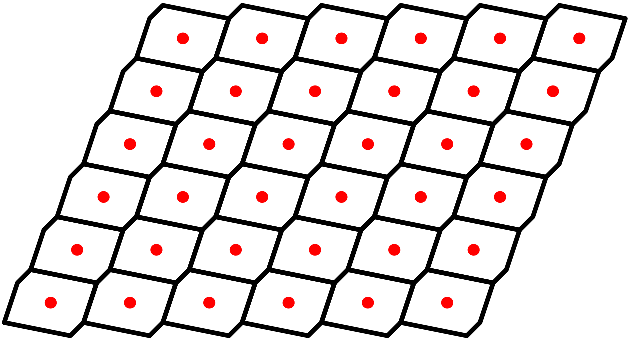


Parallelohedra and Lattices

Let $\mathcal{T}_P$ be the unique face-to-face tiling of $\mathbb{R}^d$ into parallel copies of P. Then centers of tiles forms a lattice Λ_P.



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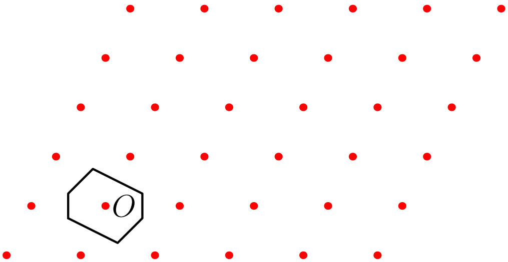

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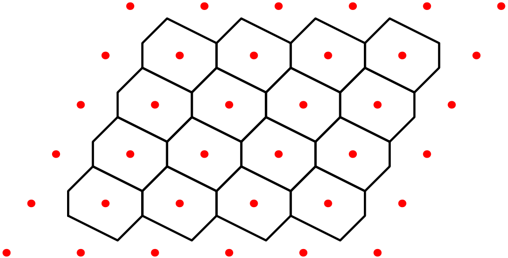
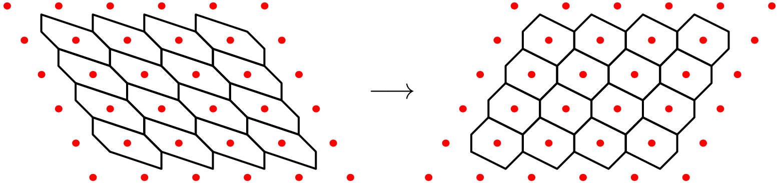
- Then DV_Λ is a parallelohedron and points of Λ are centers of correspondent tiles.



Voronoi conjecture

Conjecture (G.Voronoi, 1909)

Every parallelohedron is affine equivalent to Dirichlet-Voronoi polytope of some lattice Λ.



Some known results

Definition

A d-dimensional parallelohedron P is called **primitive** if every vertex of the corresponding tiling belongs to exactly $d+1$ copies of P.

Theorem (G.Voronoi, 1909)

The Voronoi conjecture is true for primitive parallelohedra.

Some known results

Definition

A d-dimensional parallelohedron P is called **k-primitive** if every k-face of the corresponding tiling belongs to exactly $d+1-k$ copies of P.

Theorem (O.Zhitomirskii, 1929)

The Voronoi conjecture is true for $(d-2)$-primitive d-dimensional parallelohedra. Or the same, it is true for parallelohedra without belts of length 4.

Dual cells

Definition

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Theorem (A.Magazinov, 2013)

Dual k-cell has at most 2^k vertices.

Dual 3-cells and 4-dimensional parallelohedra

Lemma (B.Delone, 1929)

There are five types of three-dimensional dual cells: tetrahedron, octahedron, quadrangular pyramid, triangular prism and cube.

Theorem (B.Delone, 1929)

The Voronoi conjecture is true for four-dimensional parallelohedra.

Delone used this result to find full classification of four-dimensional parallelohedra. He found 51 of them and the last 52nd was added by M.Shtogrin in 1973.

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Theorem (A.Ordine, 2005)

The Voronoi conjecture is true for parallelohedra without cubical or prismatic dual 3-cells.

Equivalent Statement

Problem (Dual conjecture)

For every parallelohedral tiling $\mathcal{T}_P$ with lattice Λ there exist a positive definite quadratic form $Q(\mathbf{x}) = \mathbf{x}^T Q\mathbf{x}$ such that P is Dirichlet-Voronoi polytope of Λ with respect to metric defined by Q.

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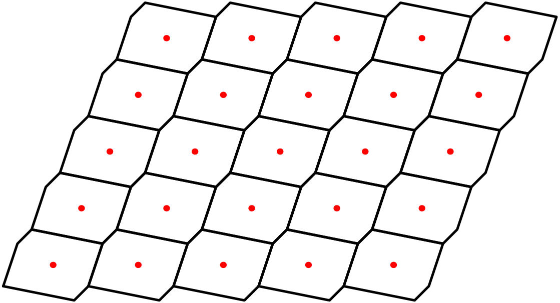
Problem

Prove that for dual tiling $\mathcal{T}_P^*$ there exist a positive definite quadratic form $Q(\mathbf{x}) = \mathbf{x}^T Q\mathbf{x}$ (or an ellipsoid E that represents a unit sphere with respect to Q) such that $\mathcal{T}_P^*$ is a Delone tiling with respect to Q and centers of corresponding empty ellipsoids are in vertices of tiling $\mathcal{T}_P$

Equivalent Statement II



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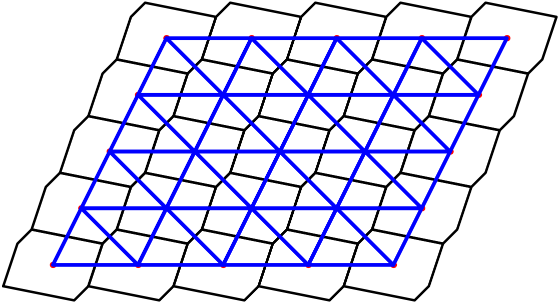

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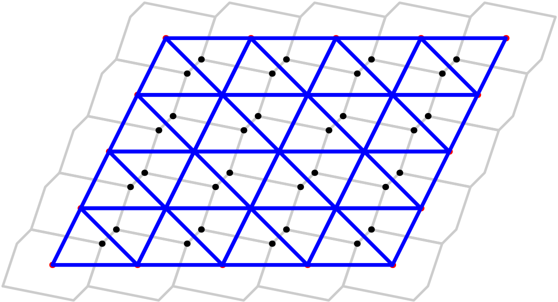

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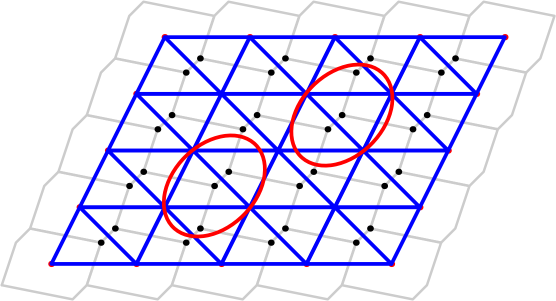

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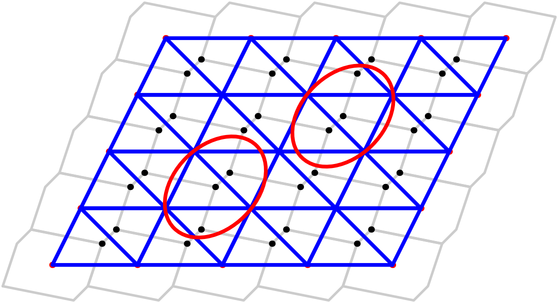

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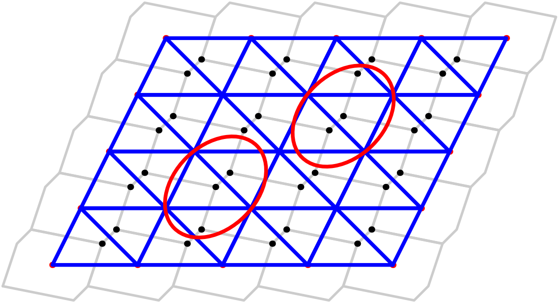

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Several more equivalent reformulations can be found in work of M.Deza and V.Grishukhin "Properties of parallelotopes equivalent to Voronoi's conjecture", 2004.

Canonical scaling

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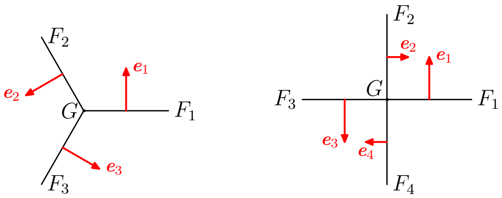

$$\sum \pm n(F_i)\mathbf{e}_i = \mathbf{0}$$

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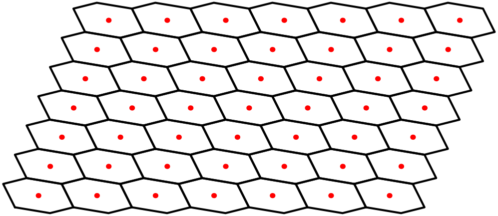
- If facets F_1 and F_2 are opposite in one parallelohedron then values of canonical scaling on F_1 and F_2 are equal.

Voronoi's Generatrissa



Consider we have a canonical scaling defined on tiling $\mathcal{T}_P$.

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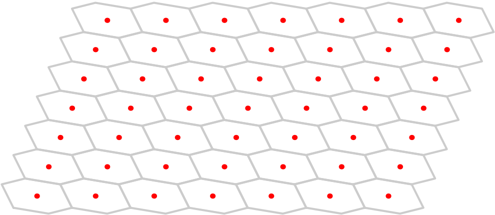

We will construct a piecewise linear generatrissa function $\mathcal{G}:\mathbb{R}^d\longrightarrow\mathbb{R}$.

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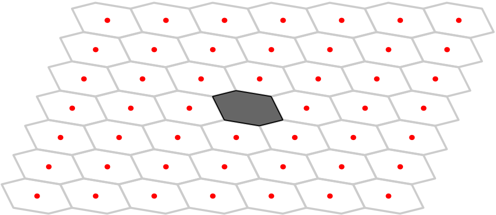

Step 1: Put $\mathcal{G}$ equal to 0 on one of tiles.

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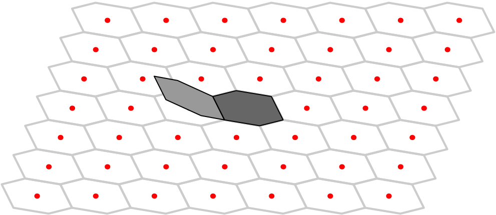

Step 2: When we pass through one facet of tiling the gradient of $\mathcal{G}$ changes accordingly to canonical scaling.

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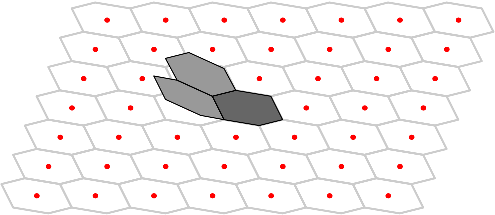
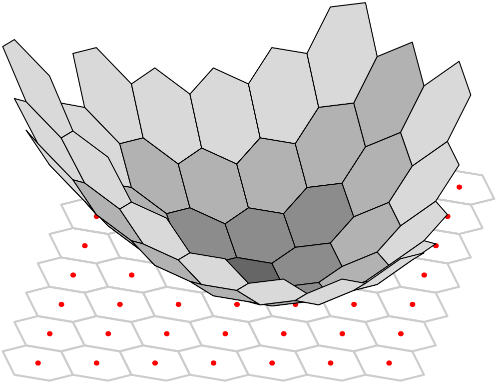


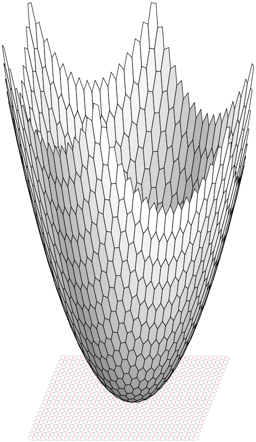
Step 2: Namely, if we pass a facet F with normal vector $\mathbf{e}$ then we add vector $n(F)\mathbf{e}$ to gradient.

Voronoi's Generatrissa



We obtain a graph of generatrissa function $\mathcal{G}$.

Voronoi's Generatrissa II



What does this graph looks like?

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So to prove the Voronoi conjecture it is sufficient to construct a canonical scaling on the tiling $\mathcal{T}_P$.
Works of Voronoi, Zhitomirskii and Ordine based on this approach.

Necessity of Generatrissa

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Tangents to parabola in points A and B intersects in the "midpoint" of AB.

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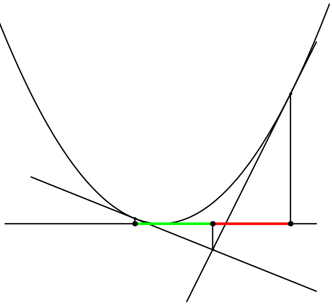

Necessity of Generatrissa

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This lemma leads to the "usual" way of constructing the Voronoi diagram for a given point set.

- We lift points onto paraboloid $y = x^T x$ in $\mathbb{R}^{d+1}$.
- Construct tangent hyperplanes.
- Take the intersection of upper-halfspaces.
- And project this polyhedron back on the initial space.

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Lemma

The Voronoi conjecture is true for P iff for any generic cycle $g(\gamma) = 1$.

Properties of gain function

Definition

Consider a manifold P_δ that is a surface of parallelohedron P with deleted closed non-primitive $(d-2)$-faces. We will call this manifold the **δ-surface** of P.

The gain function is well defined on any cycle on P_δ.

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Lemma (A.Gavrilyuk, A.G., A.Magazinov)

The gain function gives us a homomorphism

$$g : \pi_1(P_\delta) \longrightarrow \mathbb{R}_+$$

and the Voronoi conjecture is true for P iff this homomorphism is trivial.

Improvement

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- Moreover we can exclude the torsion part of the group $H_1(P_\pi, \mathbb{Z})$ since there is no torsion in the group $\mathbb{R}_+$.

Finally we get the group $H_1(P_\pi, \mathbb{Q})$.

The new result on Voronoi conjecture

Theorem (A.Gavrilyuk, A.G., A.Magazinov)

The Voronoi conjecture is true for parallelohedra with trivial group $\pi_1(P_\delta)$, i.e. for polytopes with simply connected δ-surface.

In $\mathbb{R}^3$: cube, rhombic dodecahedron and truncated octahedron.

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Theorem (A.Gavrilyuk, A.G., A.Magazinov)

The Voronoi conjecture is true for parallelohedra with trivial group $\pi_1(P_\delta)$, i.e. for polytopes with simply connected δ-surface.

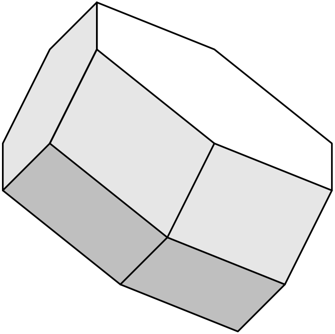
In $\mathbb{R}^3$: cube, rhombic dodecahedron and truncated octahedron.

After applying all improvements we get:

Theorem (A.Gavrilyuk, A.G., A.Magazinov)

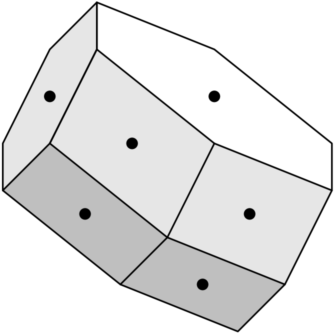
If group of one-dimensional homologies $H_1(P_\pi, \mathbb{Q})$ of the π-surface of parallelohedron P is generated by half-belt cycles then the Voronoi conjecture is true for P.

How one can apply this theorem?



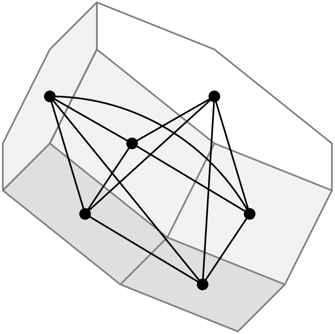
We start from a parallelohedron P.

How one can apply this theorem?



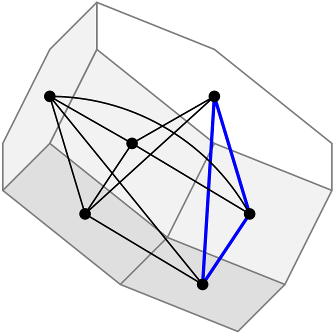
Then put a vertex of graph G for every pair of opposite facets.

How one can apply this theorem?



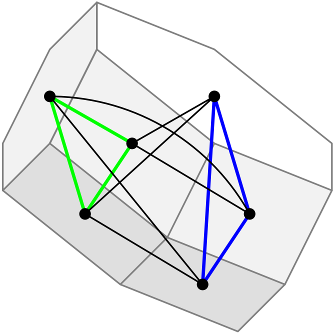
Draw edges of G between pairs of facets with common primitive $(d-2)$-face.

How one can apply this theorem?



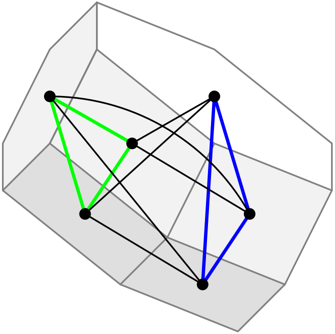
List all "basic" cycles γ that has gain function 1 for sure. These are half-belt cycles.

How one can apply this theorem?



List all "basic" cycles γ that has gain function 1 for sure. These are half-belt cycles. And trivially contractible cycles around $(d-3)$-face.

How one can apply this theorem?



Check that basic cycles generates all cycles of graph G.

Three- and four-dimensional parallelohedra

Using described algorithm we can check that every parallelohedron in $\mathbb{R}^3$ and $\mathbb{R}^4$ has homology group $H_1(P_\pi, \mathbb{Q})$ generated by half-belts cycles and therefore it satisfies our condition.

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Assume that Voronoi conjecture is true for parallelohedron P. How many there are Voronoi polytopes that are affinely equivalent to P?

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If P is primitive then there is unique Voronoi polytope equivalent to P.

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Theorem (L.Michel, S.Ryshkov, M.Senechal, 1995)

If P is primitive then there is unique Voronoi polytope equivalent to P.

Theorem (N.Dolbilin, J.-i.Itoh, C.Nara, 2011)

If graph G of P is connected then there is **at most** *one Voronoi polytope equivalent to P.*

Theorem (A.Gavrilyuk, to appear)

If G has k components then the set of Voronoi polytopes equivalent to P is either empty or a k-orbifold.

Extension of Parallelohedra

Definition

A vector v is called **free** with respect to parallelohedron P if the Minkowski sum $P + \frac{1}{2}[-v, v]$ is a parallelohedron.

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Theorem (V.Grishukhin, 2006, corrected proof in 2013 by A.Magazinov)

A vector v is free with respect to P iff v is parallel to at least one facet from every 6-belt.

Theorem (A.Magazinov, preprint)

If vector v is free with respect to Voronoi polytope P then the Voronoi conjecture is true for $P + \frac{1}{2}[-v, v]$.

THANK YOU!