Belt diameter of some class of space filling zonotopes

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Polytopes with centrally symmetric facets

Consider a family $\mathcal{P}_C$ of all d-dimensional polytopes with centrally symmetric facets.

Theorem (A.D. Alexandrov anf G.C. Shephard)

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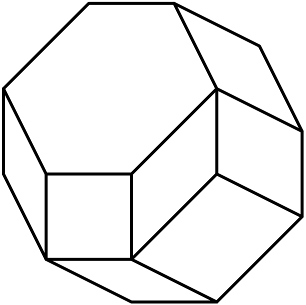
For any $(d-2)$-dimensional face F of P we can define a *belt* $\mathcal{B}_P(F)$ as a set of all facets of P parallel to F.
We can construct it by moving from one facet to another across $(d-2)$-faces parallel to F.

Constructing belts



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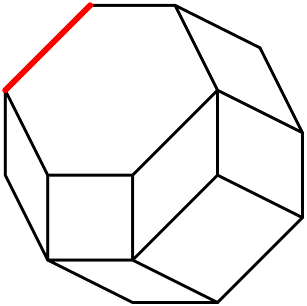

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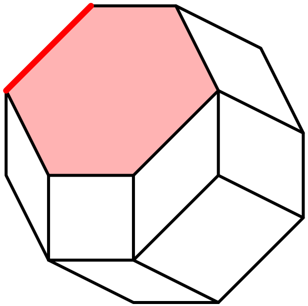

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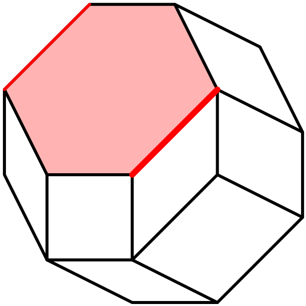

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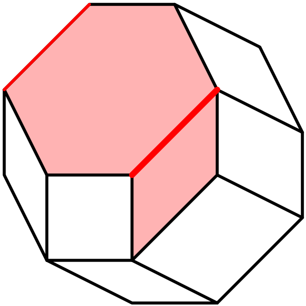

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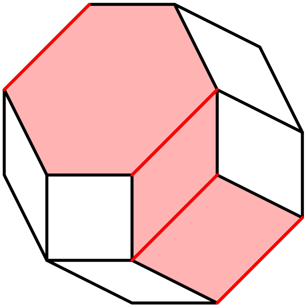

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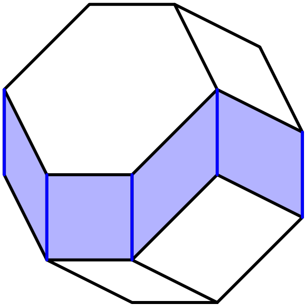

Belt distance and belt diameter

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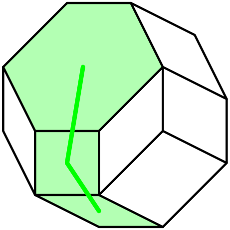
Belt distance between two facets of P is the minimal number of belts that we need to travel from one facets to another.



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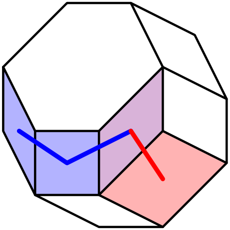

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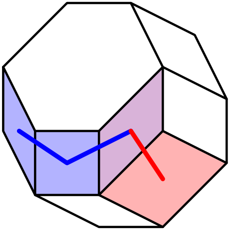

Definition

Belt diameter of P is the maximal belt distance between its facets.

Why this is interesting?

Conjecture (G.Voronoi, 1909)

Every convex polytope that tiles space with translation copies is affinely equivalent to Dirichlet-Voronoi polytope of some lattice.

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One of the most popular (and one the most successful for now) approaches to prove the Voronoi conjecture in parallelohedra theory uses a canonical scaling function.

And values of canonical scaling can be uniquely defined on facets in one belt of length equal to 6.

Questions

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We can add some restriction for maximal length of any belt or we can restrict ourselves only to certain subfamily of polytopes with centrally symmetric facets. Or both.

Small belt lengths

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  There are 5 three-dimensional parallelohedra, 52 four-dimensional and dozens of thousands of five-dimensional. And maximal belt diameter is unknown for all cases except $\mathbb{R}^3$.

Zonotopes

Definition

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The equivalent property is the following.

Theorem (P. McMullen, 1980)

For $d > 3$ a d-dimensional polytope is a zonotope if and only if all $(d-2)$-faces of P are centrally symmetric.

Questions about belt diameter of zonotopes

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Claim

If $k \geq d$ or if we do not restrict he maximal value of belt length then the answer is $d-1$ and it is sharp.

An idea how to get an upper bound

Definition

Two sets E and F of $d-1$ vectors in $\mathbb{R}^d$ each are called *conjugated* if

$$\dim(E \cup \mathbf{f}_i) = \dim(F \cup \mathbf{e}_j = d.$$

Correspondent zonotope $Z(E \cup F)$ is called *symmetric*.

Theorem (A.G.)

The maximal belt diameter for zonotope is also achieved on some symmetric zonotope of smaller or equal dimension.

A new basis for symmetric zonotopes

Lemma

For space-filling zonotopes we can find an upper bound only for zonotopes with set of zone vectors

$$V = \left(\begin{array}{c|c} E_{d-1} & A \\ \hline 0\dots0 & 1\dots1 \end{array} \right),$$

where A is a $0/1$-matrix with at least half of zeros in each row.

Results on belt diameter of zonotopes

Theorem (A.G.)

Belt diameter of d-dimensional space-filling zonotope is not greater than

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Using similar technique for arbitrary belt length we can prove

Claim

Belt diameter of d-dimensional zonotope with belt length at most k does not exceed

$$\left\lceil \log_{1+\frac{1}{k-2}} d\right\rceil$$

Π-zonotopes and their diameters

Definition

If U is a subset of the vector set $E(d) = \{\mathbf{e}_i - \mathbf{e}_j\}_{i,j=1}^{d+1}$ where $\mathbf{e}_i$ is the standard basis in $\mathbb{R}^{d+1}$ then zonotope $Z(U)$ is called a *Π-zonotope*.

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Theorem (A.G.)

Belt diameter of d-dimensional Π-zonotope is not greater than 2 if $d \leq 6$ and 3 if $d > 6$. These bounds are sharp in any dimension.

THANK YOU!