Parallelohedra and the Voronoi Conjecture

Alexey Garber

Bielefeld University
November 28, 2012

Parallelohedra

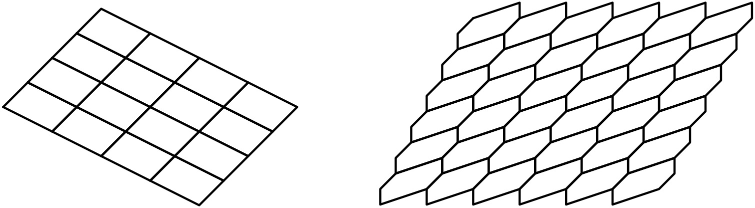
Definition

Convex d-dimensional polytope P is called a *parallelohedron* if $\mathbb{R}^d$ can be (face-to-face) tiled into parallel copies of P.

Parallelohedra

Definition

Convex d-dimensional polytope P is called a *parallelohedron* if $\mathbb{R}^d$ can be (face-to-face) tiled into parallel copies of P.



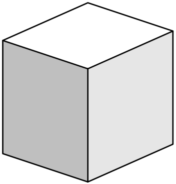
Two types of two-dimensional parallelohedra

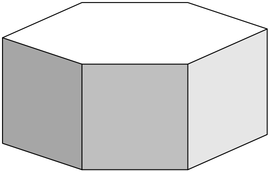
Three-dimensional parallelohedra

In 1885 Russian crystallographer E.Fedorov listed all types of three-dimensional parallelohedra.

Three-dimensional parallelohedra

In 1885 Russian crystallographer E.Fedorov listed all types of three-dimensional parallelohedra.

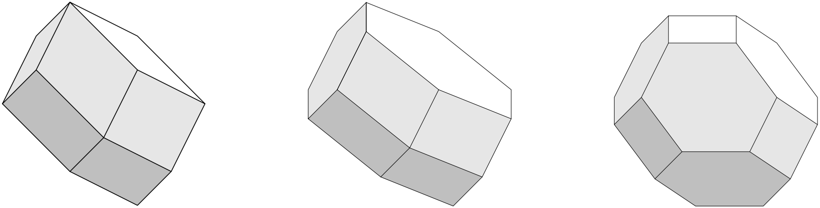




Parallelepiped and hexagonal prism with centrally symmetric base.

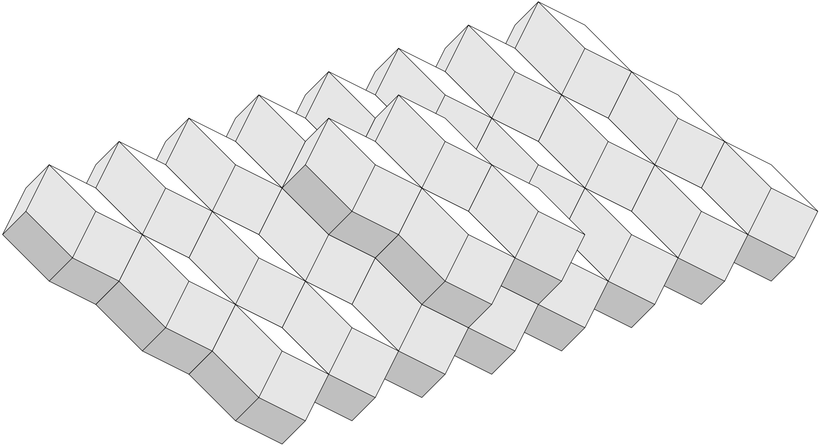
Three-dimensional parallelohedra

In 1885 Russian crystallographer E.Fedorov listed all types of three-dimensional parallelohedra.



Rhombic dodecahedron, elongated dodecahedron, and truncated octahedron

Tiling by rhombic dodecahedra



Properties of parallelohedra

Theorem (H.Minkowski, 1897)

Any d-dimensional parallelohedron P satisfies the following conditions:

1. *P is centrally symmetric;*
2. *Any facet of P is centrally symmetric;*
3. *Projection of P along any its $(d-2)$-dimensional face is parallelogram or centrally symmetric hexagon. The set of facets projected onto sides of such polygon is called a belt.*

Properties of parallelohedra

Theorem (H.Minkowski, 1897)

Any d-dimensional parallelohedron P satisfies the following conditions:

1. *P is centrally symmetric;*
2. *Any facet of P is centrally symmetric;*
3. *Projection of P along any its $(d-2)$-dimensional face is parallelogram or centrally symmetric hexagon. The set of facets projected onto sides of such polygon is called a belt.*

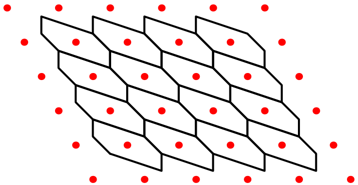
Theorem (B.Venkov, 1954)

Minkowski conditions are **sufficient** *for convex polytope P to be a parallelohedron.*

Voronoi conjecture

Conjecture (G.Voronoi, 1909)

Every parallelohedron is affine equivalent to Dirichlet-Voronoi polytope of some lattice Λ.



Known results

Theorem (G.Voronoi, 1909)

The Voronoi conjecture is true for primitive parallelohedra.

Theorem (O.Zhitomirskii, 1929)

The Voronoi conjecture is true for $(d-2)$-primitive d-dimensional parallelohedra. Or the same, it is true for parallelohedra without belts of length 4.

Theorem (R.Erdahl, 1999)

The Voronoi conjecture is true for zonotopes.

Dual cells

Definition

The *dual cell* for a face F of given parallelohedral tiling is the set of all centers of parallelohedra that shares F. If F is $(d-k)$-dimensional then the correspondent cell is called *k-cell*.

Dual cells

Definition

The *dual cell* for a face F of given parallelohedral tiling is the set of all centers of parallelohedra that shares F. If F is $(d-k)$-dimensional then the correspondent cell is called *k-cell*.

The set of all dual cells of the tiling with correspondent incidence relation determines a structure of a cell complex.

Dual cells

Definition

The *dual cell* for a face F of given parallelohedral tiling is the set of all centers of parallelohedra that shares F. If F is $(d-k)$-dimensional then the correspondent cell is called *k-cell*.

The set of all dual cells of the tiling with correspondent incidence relation determines a structure of a cell complex.

Conjecture (Dimension conjecture)

The dimension of dual k-cell is equal to k.

The dimension conjecture is necessary for the Voronoi conjecture.

Dual 3-cells and 4-dimensional parallelohedra

There are five combinatorial types of three-dimensional dual cells: tetrahedron, octahedron, quadrangular pyramid, triangular prism and cube.

Theorem (A.Ordine, 2005)

The Voronoi conjecture is true for parallelohedra without cubical or prismatic dual 3-cells.

Equivalent Statement

Problem (Dual conjecture)

For every parallelohedral tiling $\mathcal{T}_P$ with lattice Λ there exist a positive definite quadratic form $Q(\mathbf{x}) = \mathbf{x}^T Q\mathbf{x}$ such that P is Dirichlet-Voronoi polytope of Λ with respect to metric defined by Q.

Equivalent Statement

Problem (Dual conjecture)

For every parallelohedral tiling $\mathcal{T}_P$ with lattice Λ there exist a positive definite quadratic form $Q(\mathbf{x}) = \mathbf{x}^T Q\mathbf{x}$ such that P is Dirichlet-Voronoi polytope of Λ with respect to metric defined by Q.

Consider the dual tiling $\mathcal{T}_P^*$.

Equivalent Statement

Problem (Dual conjecture)

For every parallelohedral tiling $\mathcal{T}_P$ with lattice Λ there exist a positive definite quadratic form $Q(\mathbf{x}) = \mathbf{x}^T Q\mathbf{x}$ such that P is Dirichlet-Voronoi polytope of Λ with respect to metric defined by Q.

Consider the dual tiling $\mathcal{T}_P^*$.This tiling after appropriate affine transformation must be the Delone tiling of image of lattice Λ.

Equivalent Statement

Problem (Dual conjecture)

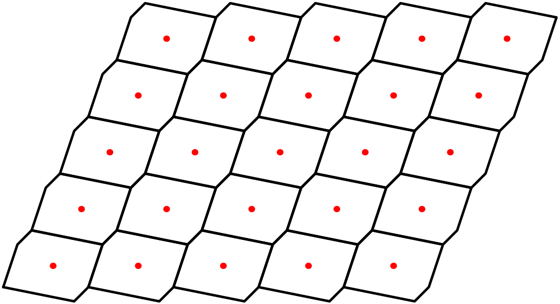
For every parallelohedral tiling $\mathcal{T}_P$ with lattice Λ there exist a positive definite quadratic form $Q(\mathbf{x}) = \mathbf{x}^T Q\mathbf{x}$ such that P is Dirichlet-Voronoi polytope of Λ with respect to metric defined by Q.

Consider the dual tiling $\mathcal{T}_P^*$.This tiling after appropriate affine transformation must be the Delone tiling of image of lattice Λ.

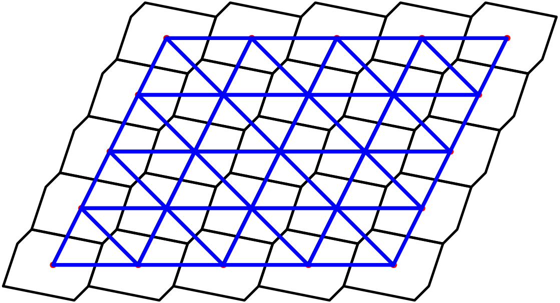
Problem

Prove that for dual tiling $\mathcal{T}_P^*$ there exist a positive definite quadratic form $Q(\mathbf{x}) = \mathbf{x}^T Q\mathbf{x}$ (or an ellipsoid E that represents a unit sphere with respect to Q) such that $\mathcal{T}_P^*$ is a Delone tiling with respect to Q and centers of correspondent empty ellipsoids are in vertices of tiling $\mathcal{T}_P$

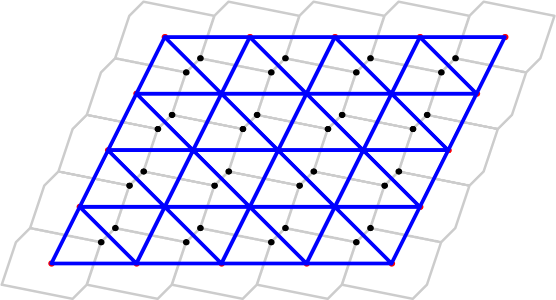
Equivalent Statement II



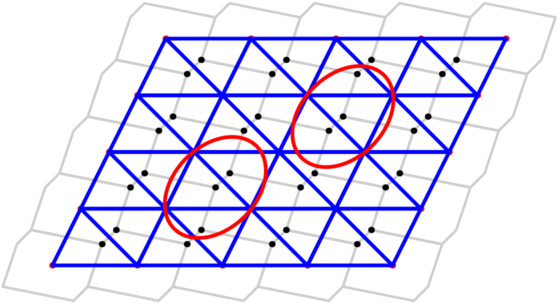
Equivalent Statement II



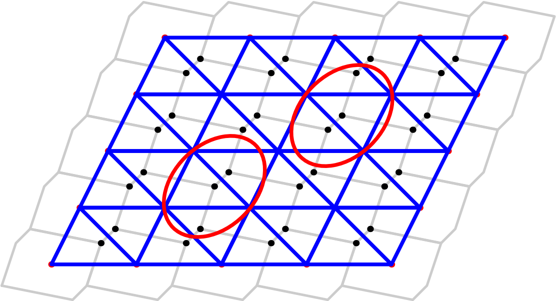
Equivalent Statement II



Equivalent Statement II

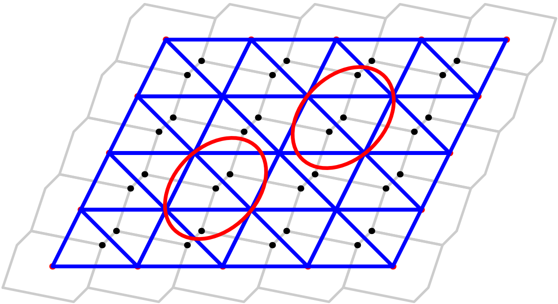


Equivalent Statement II



This approach was used by Erdahl for zonotopes.

Equivalent Statement II



This approach was used by Erdahl for zonotopes.

Several more equivalent reformulations can be found in work of Deza and Grishukhin "Properties of parallelotopes equivalent to Voronoi's conjecture", 2003.

Canonical scaling

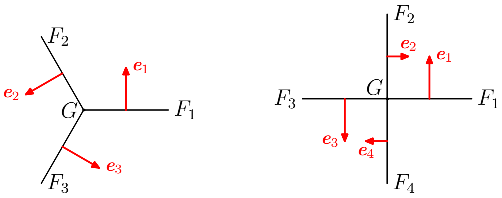
Definition

A (positive) real-valued function $n(F)$ defined on set of all facets of tiling is called *canonical scaling* if it satisfies the following conditions for facets F_i that contains arbitrary $(d-2)$-face G:

Canonical scaling

Definition

A (positive) real-valued function $n(F)$ defined on set of all facets of tiling is called *canonical scaling* if it satisfies the following conditions for facets F_i that contains arbitrary $(d-2)$-face G:



$$\sum \pm n(F_i)\mathbf{e}_i = \mathbf{0}$$

Constructing canonical scaling

How to construct a canonical scaling for a given tiling $\mathcal{T}_P$?

Constructing canonical scaling

How to construct a canonical scaling for a given tiling $\mathcal{T}_P$?

- If two facets F_1 and F_2 of tiling has a common $(d-2)$-face from 6-belt then value of canonical scaling on F_1 uniquely defines value on F_2 and vice versa.

Constructing canonical scaling

How to construct a canonical scaling for a given tiling $\mathcal{T}_P$?

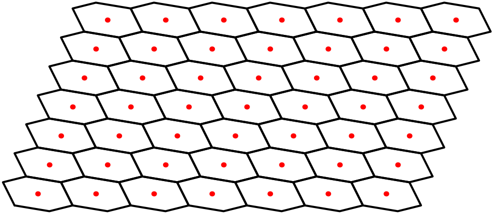
- If two facets F_1 and F_2 of tiling has a common $(d-2)$-face from 6-belt then value of canonical scaling on F_1 uniquely defines value on F_2 and vice versa.
- If facets F_1 and F_2 has a common $(d-2)$-face from 4-belt then the only condition is that if these facets are opposite then values of canonical scaling on F_1 and F_2 are equal.

Constructing canonical scaling

How to construct a canonical scaling for a given tiling $\mathcal{T}_P$?

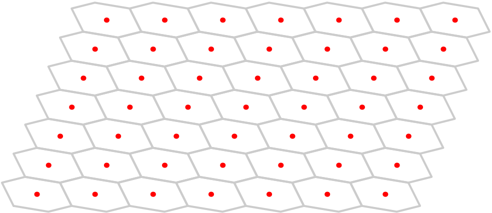
- If two facets F_1 and F_2 of tiling has a common $(d-2)$-face from 6-belt then value of canonical scaling on F_1 uniquely defines value on F_2 and vice versa.
- If facets F_1 and F_2 has a common $(d-2)$-face from 4-belt then the only condition is that if these facets are opposite then values of canonical scaling on F_1 and F_2 are equal.
- If facets F_1 and F_2 are opposite in one parallelohedron then values of canonical scaling on F_1 and F_2 are equal.

Voronoi's Generatrissa



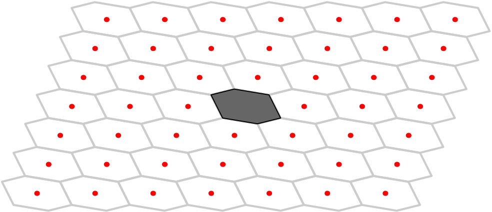
Consider we have a canonical scaling defined on tiling $\mathcal{T}_P$.

Voronoi's Generatrissa



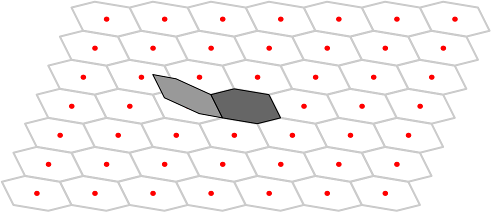
We will construct a piecewise linear generatrissa function $\mathcal{G}:\mathbb{R}^d\longrightarrow\mathbb{R}$.

Voronoi's Generatrissa



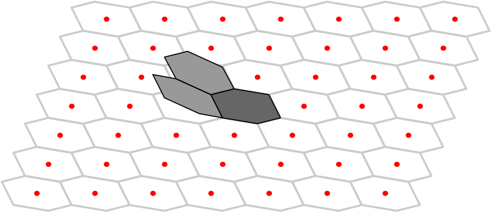
Step 1: Put $\mathcal{G}$ as 0 on one of tiles.

Voronoi's Generatrissa



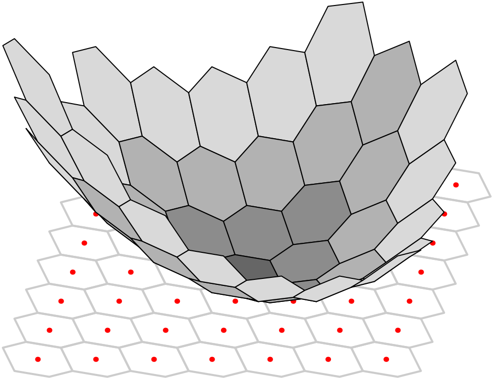
Step 2: When we pass through one facet of tiling the gradient of $\mathcal{G}$ changes accordingly to canonical scaling.

Voronoi's Generatrissa



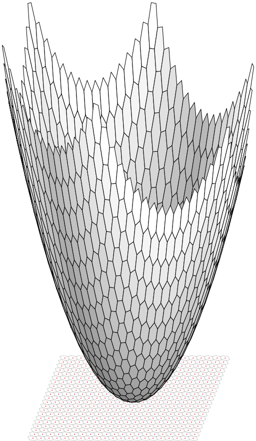
Step 2: Namely, if we pass a facet F with normal vector $\mathbf{e}$ then we add vector $n(F)\mathbf{e}$ to gradient.

Voronoi's Generatrissa



We obtain a graph of generatrissa function $\mathcal{G}$.

Voronoi's Generatrissa II



What does this graph looks like?

Properties of Generatrissa

- The graph of generatrissa $\mathcal{G}$ looks like "piecewise linear" paraboloid.

Properties of Generatrissa

- The graph of generatrissa $\mathcal{G}$ looks like "piecewise linear" paraboloid.
- And actually there is a paraboloid $y = \mathbf{x}^T Q\mathbf{x}$ for some positive definite quadratic form Q tangent to generatrissa in centers of its shells.

Properties of Generatrissa

- The graph of generatrissa $\mathcal{G}$ looks like "piecewise linear" paraboloid.
- And actually there is a paraboloid $y = \mathbf{x}^T Q\mathbf{x}$ for some positive definite quadratic form Q tangent to generatrissa in centers of its shells.
- Moreover, if we consider an affine transformation $\mathcal{A}$ of this paraboloid into paraboloid $y = \mathbf{x}^T\mathbf{x}$ then tiling $\mathcal{T}_P$ will transform into Voronoi tiling for some lattice.

Properties of Generatrissa

- The graph of generatrissa $\mathcal{G}$ looks like "piecewise linear" paraboloid.
- And actually there is a paraboloid $y = \mathbf{x}^T Q\mathbf{x}$ for some positive definite quadratic form Q tangent to generatrissa in centers of its shells.
- Moreover, if we consider an affine transformation $\mathcal{A}$ of this paraboloid into paraboloid $y = \mathbf{x}^T\mathbf{x}$ then tiling $\mathcal{T}_P$ will transform into Voronoi tiling for some lattice.

So to prove the Voronoi conjecture it is sufficient to construct a canonical scaling on the tiling $\mathcal{T}_P$.
Works of Voronoi, Zhitomirskii and Ordine based on this approach.

Necessity of Generatrissa

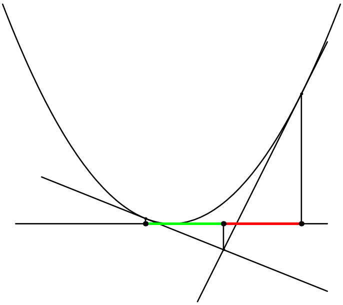
Lemma

Tangents to parabola in points A and B intersects in the "midpoint" of AB.

Necessity of Generatrissa

Lemma

Tangents to parabola in points A and B intersects in the "midpoint" of AB.



Necessity of Generatrissa

Lemma

Tangents to parabola in points A and B intersects in the "midpoint" of AB.

This lemma leads to the "usual" way of constructing the Voronoi diagram for a given point set.

- We lift points onto paraboloid $y = x^T x$ in $\mathbb{R}^{d+1}$.
- Construct tangent hyperplanes.
- Take the intersection of upper-halfspaces.
- And project this polyhedron back on the initial space.

Gain function instead of canonical scaling

We know how canonical scaling should change when we pass from one facet to neighbor facet across primitive $(d-2)$-facet F.

Gain function instead of canonical scaling

We know how canonical scaling should change when we pass from one facet to neighbor facet across primitive $(d-2)$-facet F.

Definition

We will call the multiple of canonical scaling that we achieve by passing across F the *gain function* g on F.

For any generic curve γ on surface of P that do not cross non-primitive $(d-2)$-faces we can define the value $g(\gamma)$.

Gain function instead of canonical scaling

We know how canonical scaling should change when we pass from one facet to neighbor facet across primitive $(d-2)$-facet F.

Definition

We will call the multiple of canonical scaling that we achieve by passing across F the *gain function* g on F.

For any generic curve γ on surface of P that do not cross non-primitive $(d-2)$-faces we can define the value $g(\gamma)$.

Lemma

The Voronoi conjecture is true for P iff for any generic cycle $g(\gamma)=1$.

Properties of gain function

Definition

Consider a manifold P_δ that is a surface of parallelohedron P with deleted closed non-primitive $(d-2)$-faces. We will call this manifold the *δ-surface* of P.

The gain function is well defined on any cycle on P_δ.

Properties of gain function

Definition

Consider a manifold P_δ that is a surface of parallelohedron P with deleted closed non-primitive $(d-2)$-faces. We will call this manifold the *δ-surface* of P.

The gain function is well defined on any cycle on P_δ.

Lemma (A.Gavrilyuk, A.G., A.Magazinov)

The gain function gives us a homomorphism

$$g : \pi_1(P_\delta) \longrightarrow \mathbb{R}_+$$

and the Voronoi conjecture is true for P iff this homomorphism is trivial.

Improvement

- It is easy to see that values of canonical scaling should be equal on opposite facets of P. So we can consider a *π-surface* of P that obtained from P_δ by gluing its opposite points.

Improvement

- It is easy to see that values of canonical scaling should be equal on opposite facets of P. So we can consider a *π-surface* of P that obtained from P_δ by gluing its opposite points.
- We already know some cycles (*half-belt cycles*) on P_π that g maps into 1. For example, any cycle formed by three facets F_1, F_2, F_3 that are parallel to primitive $(d-2)$-dimensional face G (like three consecutive sides of a hexagon).

Improvement

- It is easy to see that values of canonical scaling should be equal on opposite facets of P. So we can consider a *π-surface* of P that obtained from P_δ by gluing its opposite points.
- We already know some cycles (*half-belt cycles*) on P_π that g maps into 1. For example, any cycle formed by three facets F_1, F_2, F_3 that are parallel to primitive $(d-2)$-dimensional face G (like three consecutive sides of a hexagon).
- The group $\mathbb{R}_+$ is commutative so image of commutator subgroup $[\pi_1(P_\pi]$ is trivial. Therefore we factorize by commutator and group of one-dimensional homologies over $\mathbb{R}$ instead of fundamental group.

Improvement

- It is easy to see that values of canonical scaling should be equal on opposite facets of P. So we can consider a *π-surface* of P that obtained from P_δ by gluing its opposite points.
- We already know some cycles (*half-belt cycles*) on P_π that g maps into 1. For example, any cycle formed by three facets F_1, F_2, F_3 that are parallel to primitive $(d-2)$-dimensional face G (like three consecutive sides of a hexagon).
- The group $\mathbb{R}_+$ is commutative so image of commutator subgroup $[\pi_1(P_\pi]$ is trivial. Therefore we factorize by commutator and group of one-dimensional homologies over $\mathbb{R}$ instead of fundamental group.
- Moreover we can exclude the torsion part of the group $H_1(P_\pi, \mathbb{R})$ since there is no torsion in the group $\mathbb{R}_+$.

Improvement

- It is easy to see that values of canonical scaling should be equal on opposite facets of P. So we can consider a *π-surface* of P that obtained from P_δ by gluing its opposite points.
- We already know some cycles (*half-belt cycles*) on P_π that g maps into 1. For example, any cycle formed by three facets F_1, F_2, F_3 that are parallel to primitive $(d-2)$-dimensional face G (like three consecutive sides of a hexagon).
- The group $\mathbb{R}_+$ is commutative so image of commutator subgroup $[\pi_1(P_\pi]$ is trivial. Therefore we factorize by commutator and group of one-dimensional homologies over $\mathbb{R}$ instead of fundamental group.
- Moreover we can exclude the torsion part of the group $H_1(P_\pi, \mathbb{R})$ since there is no torsion in the group $\mathbb{R}_+$.

Finally we get the group $H_1(P_\pi, \mathbb{Q})$.

The new result on Voronoi conjecture

Theorem (A.Gavrilyuk, A.G., A.Magazinov)

The Voronoi conjecture is true for parallelohedra with trivial group $\pi_1(P_\delta)$, i.e. for polytopes with simply connected δ-surface.

In $\mathbb{R}^3$: cube, rhombic dodecahedron and truncated octahedron.

The new result on Voronoi conjecture

Theorem (A.Gavrilyuk, A.G., A.Magazinov)

The Voronoi conjecture is true for parallelohedra with trivial group $\pi_1(P_\delta)$, i.e. for polytopes with simply connected δ-surface.

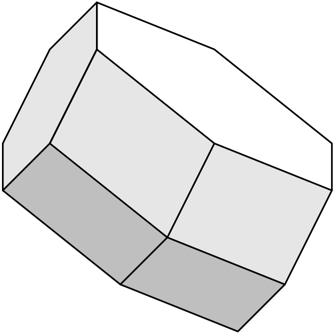
In $\mathbb{R}^3$: cube, rhombic dodecahedron and truncated octahedron.

After applying all improvements we get:

Theorem (A.Gavrilyuk, A.G., A.Magazinov)

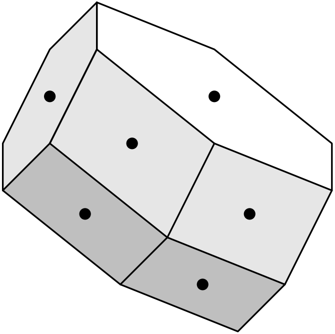
If group of one-dimensional homologies $H_1(P_\pi, \mathbb{Q})$ of the π-surface of parallelohedron P is generated by half-belt cycles then the Voronoi conjecture is true for P.

How one can apply this theorem?



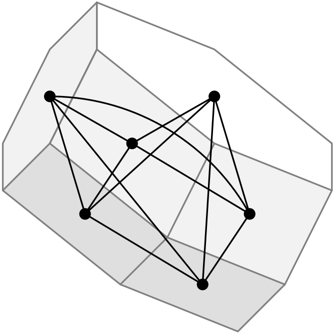
We start from a parallelohedron P.

How one can apply this theorem?



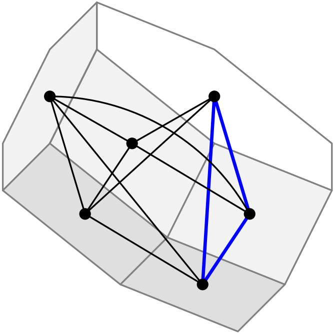
Then put a vertex of graph G for every pair of opposite facets.

How one can apply this theorem?



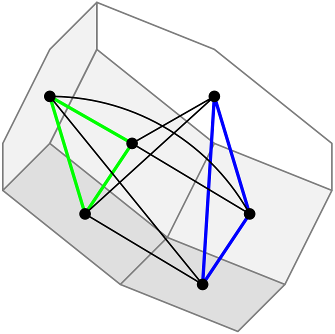
Draw edges of G between pairs of facets with common primitive $(d-2)$-face.

How one can apply this theorem?



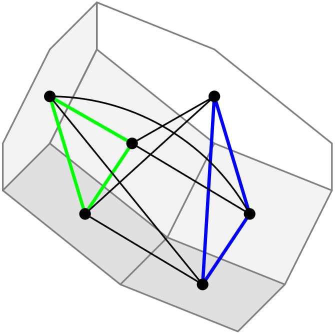
List all "basic" cycles γ that has gain function 1 for sure. These are half-belt cycles.

How one can apply this theorem?



List all “basic” cycles γ that has gain function 1 for sure. These are half-belt cycles. And trivially contractible cycles around $(d-3)$-face.

How one can apply this theorem?



Check that basic cycles generates all cycles of graph G.

Three- and four-dimensional parallelohedra

Using described algorithm we can check that every parallelohedron in $\mathbb{R}^3$ and $\mathbb{R}^4$ has homology group $H_1(P_\pi, \mathbb{Q})$ generated by half-belts cycles and therefore it satisfies our condition.

Three- and four-dimensional parallelohedra

Using described algorithm we can check that every parallelohedron in $\mathbb{R}^3$ and $\mathbb{R}^4$ has homology group $H_1(P_\pi, \mathbb{Q})$ generated by half-belts cycles and therefore it satisfies our condition.

THANK YOU!