Proof of the Voronoi conjecture on parallelotopes in a new special case

by

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Abstract

This thesis solves an important part of a long-standing problem in the theory of tilings, the Voronoi conjecture on parallelotopes.

Parallelotopes are convex polytopes which tile the Euclidean space by their translated copies, like in the honeycomb arrangement of hexagons in the plane. An important example of parallelotope is the Dirichlet-Voronoi domain for a translation lattice. For each point λ in a translation lattice, we define its Dirichlet-Voronoi (DV) domain to be the set of points in the space which are at least as close to λ as to any other lattice point.

The Voronoi conjecture, formulated by the great Ukrainian mathematician George Voronoi in 1908, states that any parallelotope is affinely equivalent to the DV-domain for some lattice.

Our work proves the Voronoi conjecture for 3-irreducible parallelotope tilings of arbitrary dimension. This result generalizes a theorem of Zhitomirskii (1927), and suggests a way to solve the conjecture in the general case.

In a separate result, we introduce the Venkov graph of a parallelotope and give a criterion for the reducibility of a parallelotope into direct Minkowski sum of two parallelotopes of smaller dimensions.

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Statement of originality

I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline. I acknowledge the helpful guidance and support of my supervisor, Professor R.M.Erdahl.

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Chapter 1

Introduction

1.1 Voronoi's conjecture on parallelotopes

This dissertation investigates parallelotopes, convex polytopes which tile Euclidean space by parallel copies. For instance, a parallelotope in the plane is either a parallelogram or a centrally symmetric hexagon.

An important example of a parallelotope is given by a Dirichlet-Voronoi domain. For each point λ in a translation lattice, we define its Dirichlet-Voronoi (DV) domain to be the set of points in the space which are at least as close to λ as to any other lattice point. A DV-domain is a centrally symmetric polytope. Since DV-domains obviously tile the space, and are all equal up to translation, they are parallelotopes. Figure 1.1 shows an example of a Dirichlet-Voronoi domain in the plane.

George Voronoi, the great Ukrainian mathematician, conjectured in 1908 ([Vor09]) that any parallelotope is a DV-domain for some lattice, or is affinely equivalent to one. This is a strong and fascinating conjecture since it gives an analytical description to a combinatorial object. Voronoi proved it for primitive tilings of d-dimensional space,

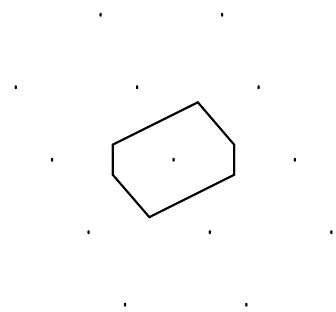


Figure 1.1: A Dirichlet-Voronoi domain

that is tilings where exactly $d+1$ parallelotopes meet at each vertex.

Since then, the conjecture has been established in a number of special cases. Zhitomirskii generalized Voronoi's argument to tilings of d-dimensional space where 3 parallelotopes meet at each $(d-2)$-face (1927, [Zhi27]), Boris Delaunay proved the conjecture for tilings of 4-dimensional space (1929, [Del29]). Robert Erdahl proved it for space filling zonotopes (1999, [Erd99]). Delaunay challenged Soviet mathematicians to resolve the conjecture in his afterword to the Russian translation of Voronoi's paper [Vor09]. We tried to answer the challenge and obtained a proof of the conjecture in a new special case which naturally generalizes Voronoi's and Zhitomirskii's work. The next section states our results.

Theoretical and practical applications of parallelotopes are numerous. Parallelotopes in dimension 3 are of great importance in crystallography. They were first classified by Evgraf Fedorov, the famous geologist and crystallographer, in 1885 ([Fed85]). Parallelotopes are also used in vector quantization (digitalization of vector data). For example, the parallelotope with 14 facets shown in figure 1.2 (rightmost) is used to encode the three-component signal vector in video monitors. In geometry of numbers, parallelotopes are applied to the study of arithmetic properties of positive definite

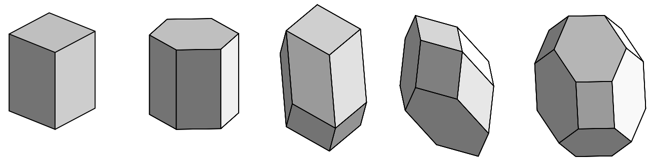


Figure 1.2: 3-dimensional parallelotopes

quadratic forms.

1.2 Our results

The idea behind our work is to investigate reducibility properties of parallelotopes.

A parallelotope is called *reducible* if it can be represented as a Minkowski sum of two parallelotopes of smaller dimensions which belong to complementary affine spaces (a direct Minkowski sum). In dimension 3, the reducible parallelotopes are the hexagonal prism and the parallelepiped (see figure 1.2). Note that the set $\mathcal{N}$ of normal vectors to facets of a reducible parallelotope can be broken down into nonempty subsets $\mathcal{N}_1$ and $\mathcal{N}_2$ so that the linear spaces $\mathrm{lin}(\mathcal{N}_1)$ and $\mathrm{lin}(\mathcal{N}_2)$ are complementary:

$$\mathbb{R}^d = \mathrm{lin}(\mathcal{N}_1) \oplus \mathrm{lin}(\mathcal{N}_2) \tag{1.1}$$

Consequently, if the Voronoi conjecture is true for irreducible parallelotopes, then it is true in all cases. But reducibility is a very restrictive property. Can we relax it?

We define the *k-reducibility* of a normal [1] parallelotope tiling of the Euclidean

[1]In a *normal*, or *face-to-face* tiling two parallelotopes intersect over a common face or do not intersect at all. This definition excludes tilings such as the brick wall pattern, where rectangles may share only a part of an edge.

d-dimensional space as follows.

Let F^{d-k} be a face of the tiling, $k > 1$. Consider the set $\mathcal{N}_{F^{d-k}}$ of normal vectors to facets of the tiling which contain F^{d-k} in their boundaries. If $\mathcal{N}_{F^{d-k}}$ can be broken down into nonempty subsets $\mathcal{N}^1_{F^{d-k}}$ and $\mathcal{N}^2_{F^{d-k}}$ so that

$$\mathrm{lin}(\mathcal{N}_{F^{d-k}}) = \mathrm{lin}(\mathcal{N}^1_{F^{d-k}}) \oplus \mathrm{lin}(\mathcal{N}^2_{F^{d-k}}), \tag{1.2}$$

then the tiling is called *locally reducible* at face F^{d-k}. If the tiling is locally reducible at one or more faces F^{d-k}, then it is called *k-reducible*.

A normal tiling which is not k-reducible is called k-irreducible. If a tiling is k-reducible, then it is m-reducible for all $1 < m < k$.

The Voronoi conjecture for 2-irreducible tilings was proved by Zhitomirskii in 1927 ([Zhi27]). Our main result is the following theorem:

Theorem 1 *The Voronoi conjecture is true for 3-irreducible parallelotope tilings.*

This is a natural generalization of Zhitomirskii's result. Before this work started, Sergei Ryshkov and Konstantin Rybnikov considered the Voronoi conjecture for 3-irreducible tilings. They tried to apply the methods of algebraic topology [Ryb05].

We can consider k-irreducible tilings, $k = 4, 5, \ldots, d$ and try to prove the Voronoi conjecture for them. We speculate it can be a way to attack the Voronoi conjecture in the general case. The conclusion of the thesis has more on this topic.

Theorem 1 deals with "local" irreducibility of parallelotope tilings. We have also investigated the "global" irreducibility of a parallelotope, using a special graph. The graph with the vertex set composed of the pairs of opposite facets of the parallelotope where

1. two distinct vertices are connected by a blue edge if the corresponding pairs of facets belong to a common 4-facet belt,

2. three distinct vertices are spanned by a red triangle if the corresponding three pairs belong to a common 6-facet belt,

is called the *Venkov graph*. A belt is a collection of facets parallel to the same $(d-2)$-face; we will see that it can contain either 4 or 6 facets. More information on belts is given in the next section. We prove the following irreducibility criterion for parallelotopes.

Theorem 2 *A parallelotope is irreducible if and only if its Venkov graph is connected by red edges.*

Examples of Venkov graphs can be found in chapter 8. Their role in parallelotope tiling theory will be discussed in the conclusion of the thesis.

1.3 Plan of the thesis

In the next chapter, we present the existing results in the parallelotope theory. The rest of the thesis is devoted to proving the results of this work, theorems 1 and 2. Knowledge of polytope theory is expected on the part of the reader; a good reference is McMullen and Shephard's book [MS71]. We also use some results and terminology of the theory of complexes; paper [Ale22] by Alexander gives enough information.

The first theorem is established in chapters 3 - 7. Chapter 3 introduces the canonical scaling and gives a tool for building it. The existence of canonical scaling is equivalent to the Voronoi conjecture. Chapter 4 defines dual cells, objects which we use to describe stars of faces in the tiling. In chapter 5 the results of the two previous chapters are applied to identify and study incoherent parallelogram dual cells, objects which shouldn't exist for the Voronoi conjecture to hold. Chapter 6 investigates polytope-theoretic properties of dual cells. It is independent from chapter 5. Finally, the main result (theorem 1) is proved in chapter 7, where all results obtained are applied to showing that no incoherent parallelogram cells can exist in a 3-irreducible tiling.

Our second result (theorem 2 on page 5) is proved in chapter 8. The proof only depends on the material in the introduction, chapters 2 and 3.

We finish with a discussion which summarizes our ideas and suggests ways to solve the Voronoi conjecture in the general case.

Chapter 2

Overview of classical results

2.1 The Minkowski and Venkov theorems

We now present the classical results in parallelotope theory. In 1897, Minkowski published his famous theorem on polytopes ([Min97]).

Theorem 3 *Let $d \geq 2$. Suppose that $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_k \in \mathbb{R}^d$ are unit vectors that span $\mathbb{R}^d$, and suppose that $\alpha_1, \ldots, \alpha_k > 0$. Then there exists a d-dimensional polytope P having external facet normals $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_k$ and corresponding facet $d-1$-volumes $\alpha_1, \ldots, \alpha_k$ if and only if*

$$\mathbf{n}_1\alpha_1 + \cdots + \mathbf{n}_k\alpha_k = 0. \tag{2.1}$$

Moreover, such a polytope is unique up to a translation.

The theorem has an important implication for parallelotopes.

Corollary 1 *(Minkowski, [Min05], 1905)*

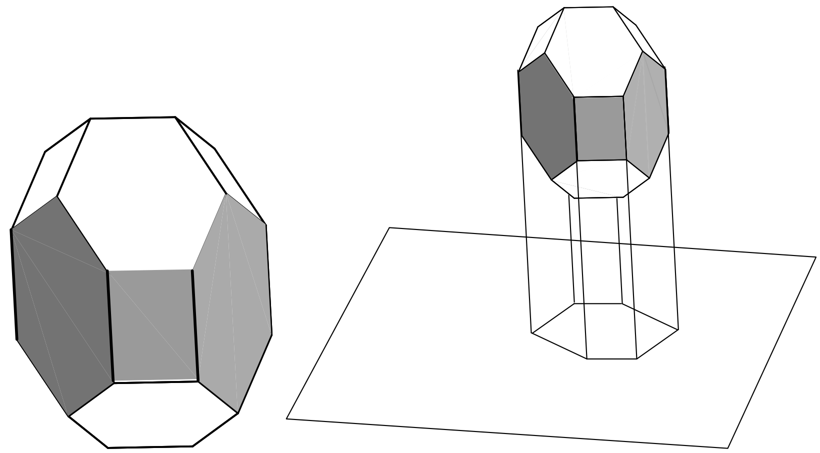


Figure 2.1: A belt and the projection along the corresponding $(d-2)$-face, $d = 3$

1. *A parallelotope is centrally symmetric* [1]

2. *Facets of a parallelotope are centrally symmetric*

3. *The projection of a parallelotope along a face of $(d-2)$ dimensions onto a complementary 2-space is either a parallelogram, or a centrally symmetric hexagon.* [2]

Theorem 4 *(Venkov, [Ven54] 1954) Conditions 1-3 are sufficient for a polytope to be a parallelotope admitting a normal (face-to-face) tiling.*

A result by Nikolai Dolbilin ([Dol00]) gives an easy way to prove this theorem.

Corollary 2 *An arbitrary parallelotope admits a normal tiling of the space.*

[1] *A set A is called* centrally symmetric *if there is a point a such that the mapping* $* : x \to 2a - x$ *maps A onto itself. Point a is called the* center of symmetry.

[2] *Properties (1) and (2) are not independent: if* $d \geq 3$*, then the central symmetry of a polytope follows from the central symmetry of its facets (ie.* $(d-1)$*-dimensional faces), see [McM76].*

Indeed, a parallelotope satisfies conditions 1-3 by Minkowski's corollary, so by Venkov's theorem it admits a normal tiling of the space by parallel copies. In the rest of the thesis, we will only consider normal tilings.

Conditions 1-3 are known as *Venkov conditions.* The second Venkov condition implies that facets of the parallelotopes are organized into *belts*: to define a belt, take a facet and a $(d-2)$ face on its boundary. Since the facet is centrally symmetric, there is a second $(d-2)$-face parallel to the first one, its symmetric copy. Now, the second $(d-2)$-face belongs to another facet of the parallelotope which is centrally symmetric as well. Its central symmetry produces a third $(d-2)$-face. Proceeding this way, we will come back to the facet we have started from. The facets that we have visited form a *belt* (see figure 2.1). When we project the parallelotope along the $(d-2)$-face, we get a parallelogram or a hexagon, by Venkov condition 3. The edges of this figure are the images of the facets in the belt; therefore, the belt contains 4 or 6 facets.

The belt and the $(d-2)$-face are called *quadruple* if the projection is a parallelogram, or *hexagonal* if the projection is a hexagon.

2.2 Stars of $(d-2)$-faces

We now discuss one important corollary of the Venkov conditions. The *star* $\mathrm{St}(F)$ of a face F of the tiling is the collection of all faces of the tiling which contain F.

Lemma 1 *([Del29]) The star of a hexagonal $(d-2)$-face contains 3 facets, where no 2 facets are parallel. The star of a quadruple $(d-2)$-face contains 4 facets. There are 2 pairs of parallel facets, but facets from different pairs are not parallel.*

The proof can be found in Delaunay's paper [Del29]. We will present the idea briefly.

Fix a parallelotope P_0 in the tiling and consider the collection $\mathcal{L}$ of parallelotopes which can be reached from P_0 by crossing facets parallel to F^{d-2}. Delaunay called $\mathcal{L}$ a "couche", or layer.

Let h be the projection along F^{d-2} onto a complementary 2-space L^2. The images $h(P)$ of parallelotopes $P \in \mathcal{L}$ form a hexagonal or parallelogram tiling of L^2, depending on whether F^{d-2} is hexagonal or quadruple. The projection gives a 1-1 correspondence between parallelotopes in $\mathcal{L}$ and the tiles in L^2. This allows for the description of the star of F^{d-2}, on the basis of the star of a vertex $h(F^{d-2})$ in the tiling of L^2.

2.3 Voronoi's work

Voronoi in paper [Vor09] proved that a primitive parallelotope is affinely equivalent to a DV-domain for some lattice (a primitive d-dimensional parallelotope produces a tiling where each vertex belongs to exactly $d+1$ parallelotopes). Since our work is based on Voronoi's ideas, we explain them now for a simple yet illustrative example, a centrally symmetric hexagon.

(A) Suppose that we have a DV-tiling of the plane by equal hexagons. The tiling has the characteristic property that the line segments connecting centers of adjacent parallelotopes are orthogonal to their common edge, like in figure 2.2. The dashed lines form a *reciprocal*, a rectilinear dual graph to the tiling.

With each edge F of the tiling, we can associate a positive number $s(F)$ equal to the length of the corresponding reciprocal edge. The numbers have the property that

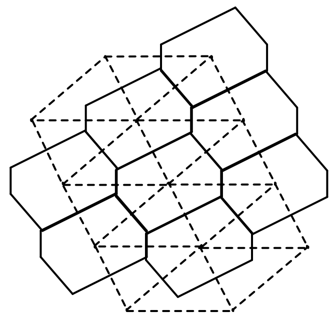


Figure 2.2: A planar DV-tiling with the reciprocal

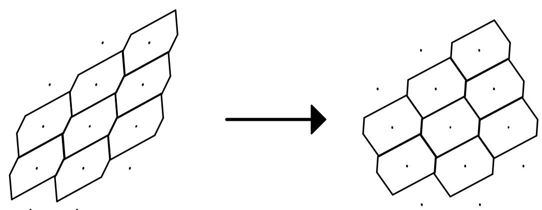


Figure 2.3: Mapping a hexagonal tiling onto a DV-tiling

for each triplet of edges F, G and H incidental to the same vertex, we have

$$s(F)\mathbf{n}_F + s(G)\mathbf{n}_G + s(H)\mathbf{n}_H = 0 \tag{2.2}$$

where $\mathbf{n}_F$, $\mathbf{n}_G$ and $\mathbf{n}_H$ are the unit normal vectors to the facets, with directions chosen appropriately (clockwise or counterclockwise). This equation simply represents the fact that the sum of edge vectors of the reciprocal triangle is equal to 0.

(B) Now, consider a tiling by an arbitrary centrally symmetric hexagon. We would like to find an affine transformation which maps the tiling onto a tiling by DV-domains, as in figure 2.3. We can still assign a positive number $s(F)$ to each edge F of the tiling, so that condition 2.2 holds for each triplet of edges incident to the same vertex.

Take an arbitrary vertex of the tiling and let e_F, e_G and e_H be the edge vectors of the three edges (pointing away from the vertex). We let $\mathbf{n}_F = R_{\pi/2}\frac{e_F}{|e_F|}$, $\mathbf{n}_G = R_{\pi/2}\frac{e_G}{|e_G|}$, $\mathbf{n}_H = R_{\pi/2}\frac{e_H}{|e_H|}$ where $R_{\pi/2}$ is the rotation by angle $\pi/2$. Vectors $\mathbf{n}_F$, $\mathbf{n}_G$ and $\mathbf{n}_H$ are normal to facets F, G and H. It it easy to see that there are unique (up to a common multiplier) positive numbers $s(F)$, $s(G)$ and $s(H)$ so that equation 2.2 holds. Then assign $s(F)$ to all edges parallel to F, $s(G)$ to all edges parallel to G, $s(H)$ to all edges parallel to H. This construction will be called a *canonical scaling* of the tiling. The numbers $s(F)$ are called *scale factors.* The choice of terminology is due to the fact that we are scaling normal vectors to edges of the tiling.

A canonical scaling allows us to find the desired affine transformation, using a wonderful method invented by Voronoi. His method involves the construction of a convex surface in 3-space, made of planar hexagons, which projects to a hexagonal tiling of the (x_1, x_2)-plane. This surface can be considered as a piecewise-planar function $z = G(x_1, x_2)$ (see figure 2.4).

We will define G in terms of its gradient vector, which will be constant on each of the hexagons. The rule of gradient assignment is as follows. We set the gradient to 0 on some fixed hexagon and let a point x travel on the plane, avoiding vertices of the tiling. *When x crosses the edge F between two adjacent hexagons, the gradient of the function G changes by $s(F)\mathbf{n}_F$, where $\mathbf{n}_F$ is the unit normal vector to the edge directed from the first hexagon to the second one, and $s(F)$ is the scale factor.*

This definition of the gradient is consistent, that is, different paths to the same hexagon result in the same gradient. To prove the consistency, we need to check that the sum of increments $s(F)\mathbf{n}_F$ along a closed circuit of hexagons is zero. This follows

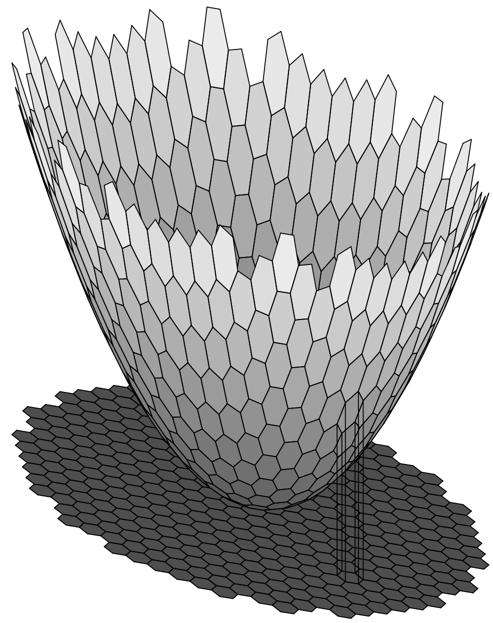
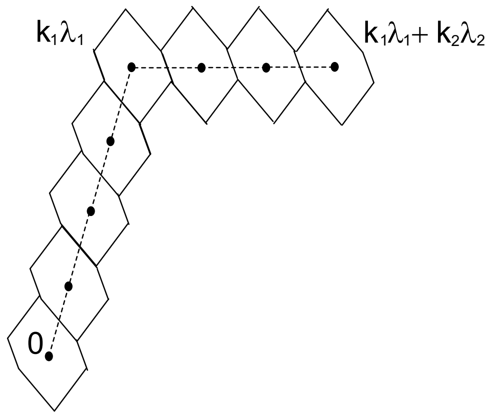


Figure 2.4: The generatrissa

from equation (2.2).

The function G was called the *generatrissa* by Voronoi. A drawing of its graph is shown in figure 2.4. It looks like a paraboloid, a graph of a positive definite quadratic form. This is not a coincidence. We can actually calculate the quadratic form whose graph is inscribed into the graph of G.

Let λ_1, λ_2 be two non-collinear *facet vectors*, translation vectors which shift the hexagon onto its neighbors. Each center of a hexagon in the tiling can be represented as a sum of vectors λ_1, λ_2 with integer coefficients. What is the value $G(x)$ of the generatrissa at the point $x = k_1\lambda_1 + k_2\lambda_2$? To calculate $G(x)$, we traverse the path between 0 and x in the following way:



We have $G(0) = 0$. To calculate the difference $G(k_1\lambda_1) - G(0)$, we integrate the gradient of the generatrissa G along the line segment $[0, k_1\lambda_1]$. Whenever we switch to the adjacent hexagon, the gradient increases by $\mathbf{n}_1$. The result is

$$G(k_1\lambda_1) - G(0) = (\lambda_1 \cdot \mathbf{n}_1)(1 + 2 + \cdots + (k_1 - 1) + \frac{k_1}{2}) = (\lambda_1 \cdot \mathbf{n}_1)\frac{k_1^2}{2}. \tag{2.3}$$

The gradient of the generatrissa at the point $k_1\lambda_1$ is equal to $k_1\mathbf{n}_1$. Next, we calculate the difference between $G(k_1\lambda_1 + k_2\lambda_2)$ and $G(k_1\lambda_1)$. Again, we integrate the gradient of G along the line segment. We start with gradient $k_1\mathbf{n}_1$ at point $k_1\lambda_1$ and note that the gradient increases by $\mathbf{n}_2$ each time we cross the boundary between hexagons. Therefore

$$\begin{aligned} G(k_1\lambda_1 + k_2\lambda_2) - G(k_1\lambda_1) &= \\ (\lambda_2 \cdot \mathbf{n}_2)(1 + 2 + \cdots + (k_2 - 1) + \frac{k_2}{2}) + (k_2\lambda_2) \cdot (k_1\mathbf{n}_1) &= \\ (\lambda_2 \cdot \mathbf{n}_2)\frac{k_2^2}{2} + (\lambda_2 \cdot \mathbf{n}_1)k_1k_2. \end{aligned} \tag{2.4}$$

Summing the two differences, we have

$$G(k_1\lambda_1 + k_2\lambda_2) = G(x) = (\lambda_1 \cdot \mathbf{n}_1)\frac{k_1^2}{2} + (\lambda_2 \cdot \mathbf{n}_2)\frac{k_2^2}{2} + (\lambda_2 \cdot \mathbf{n}_1)k_1k_2. \tag{2.5}$$

We use the symbol $Q(x)$ for the quadratic form $(\lambda_1 \cdot \mathbf{n}_1)\frac{y_1^2}{2} + (\lambda_2 \cdot \mathbf{n}_2)\frac{y_2^2}{2} + (\lambda_2 \cdot \mathbf{n}_1)y_1y_2$, where $x = y_1\lambda_1 + y_2\lambda_2$.

The two functions $G(x)$ and $Q(x)$ coincide at the centers of hexagons. Moreover, one can check that, at these points, the gradients of the functions are equal, which shows that each lifted hexagon is tangent to the graph of the function $Q(x)$. This explains why the generatrissa looks like the graph of a quadratic form.

Next, we prove that the generatrissa is a convex function. Take any two points in the plane of the tiling and connect them by a line segment. By a small shift of the points, we can assure that the line segment does not pass through the vertices of the tiling. The function G limited to the line segment is piecewise linear. Its slope is equal to the scalar product of the direction of the line segment to the gradient of the generatrissa. Each time the line segment crosses the edge, the gradient of the generatrissa changes by a normal vector to the edge being crossed, which points into the next hexagon. Therefore the slope of G increases along the line segment. This proves that G is convex. The quadratic form Q is positive definite since it is bounded by the function G from below and G is positive everywhere outsize the zero hexagon (since the six hexagons surrounding the zero hexagon have been lifted upward).

Making an affine transformation if necessary, we can assume that $Q(x) = x_1^2 + x_2^2$. The generatrissa is then formed by tangent planes at points $(\lambda, Q(\lambda))$ where λ is the center of some hexagon of the tiling.

Let λ_1, λ_2 be the centers of two adjacent hexagons. It is a fact from elementary

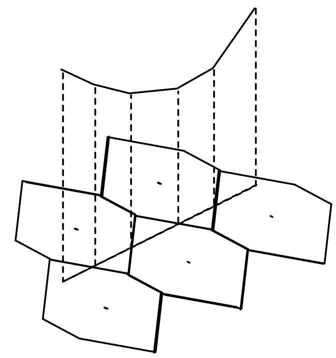


Figure 2.5: To the proof of the convexity of generatrissa

geometry that the intersection of the planes tangent to the graph of the positive definite quadratic form Q at the points $(\lambda_1, Q(\lambda_1))$ and $(\lambda_2, Q(\lambda_2))$ projects to the bisector line of λ_1 and λ_2. Therefore each hexagon in the tiling is bounded by bisectors to the line segments which connect its center to the centers of adjacent parallelotopes. This means that the hexagon is the DV-domain for the lattice of centers.

Of course, we could find a much simpler way to map an arbitrary hexagonal tiling onto a DV-tiling. However, the method just described works almost verbatim for a parallelotope tiling of arbitrary dimension, so long as the tiling can be equipped with a canonical scaling. In the next chapter, we describe a tool for obtaining it.

2.4 Space filling zonotopes and zone contraction

Reciprocal construction is one direction of research in the Voronoi conjecture. There is another direction which considers *zonotopes* and properties of parallelotopes described by *zones*. A zonotope (zonohedron) is a Minkowski sum of a finite collection of line segments. Below we review some results on zonotopes that fill the space by parallel copies.

All parallelotopes of dimensions 2 and 3 are zonotopes, but this is not true for dimension 4 and higher. However, zonotopes continue to play a prominent role in the 4-dimensional parallelotope family; Delaunay in 1929 proved that up to affine equivalence, a parallelotope of dimension 4 is either the 24-cell, a Minkowski sum of the 24-cell and a zonotope, or a space filling zonotope ([Del29]). The 24-cell is the DV-domain for the lattice $\{z \in \mathbb{Z}^4 : z_1 + \cdots + z_4 \equiv 0 \pmod 2\}$ in 4-dimensional space. Erdahl in 1999 proved the Voronoi conjecture for space filling zonotopes ([Erd99]).

Given a parallelotope, a *zone* is the set of edges parallel to a given vector. A zone is called *closed* if each 2-dimensional face of the parallelotope either contains two edges from the zone, or none. If a parallelotope has a closed zone, it can be transformed so that the lengths of edges in the zone decrease by the same amount. This operation is called *zone contraction*; the reverse operation is called *zone extension*.

Engel in [Eng00] discovered that there are at least 103769 combinatorial types of 5-dimensional parallelotopes. This is an enormous jump from just 52 combinatorial types[3] of parallelohedra of dimension 4. He uses "contraction types" of parallelotopes to refine the combinatorial type classification. Contraction type is a complex attribute of a parallelotope, defined using zone extension and zone contraction operations.

Finally we note that a direct, "brute force" computer enumeration of parallelotopes in a given dimension is hard since the combinatorial complexity of parallelotopes grows fast: the maximal number of facets of a parallelotope in dimension d is $2(2^d - 1)$.

[3] Two polytopes are of the same combinatorial type if and only if their face lattices are isomorphic. See book [MS71] for details.

Chapter 3

Canonical scaling and the quality translation theorem

We begin to prove our main result, the Voronoi conjecture for the case of 3-irreducible tilings. In this chapter we define the canonical scaling of a tiling, whose existence is equivalent to the statement of the Voronoi conjecture, and present the quality translation theorem, a tool for building the canonical scaling. As an illustration, we prove the Voronoi conjecture for the case of primitive tilings.

3.1 Canonical scaling

We have encountered the canonical scaling of a hexagonal tiling of the plane in section 2.3. It allowed us to construct an affine transformation which mapped the hexagon onto a DV-domain, thus proving the Voronoi conjecture. It turns out that the same idea works for a general normal parallelotope tiling. The definition below depends on the discussion of stars of $(d-2)$-faces on page 9.

Definition 1 *Let S be an arbitrary collection of facets in the tiling. A* canonical scaling *of S is an assignment of a positive number $s(F)$ (called a* scale factor*) to each facet $F \in S$ such that*

1. *If $F_1, F_2, F_3 \in S$ are the facets in the star of a hexagonal $(d-2)$-face, then for some choice of unit normals $\mathbf{n}_{F_i}$ to the facets*

$$s(F_1)\mathbf{n}_{F_1} + s(F_2)\mathbf{n}_{F_2} + s(F_3)\mathbf{n}_{F_3} = 0 \tag{3.1}$$

2. *If $F_1, F_2, F_3, F_4 \in S$ are the facets in the star of a quadruple $(d-2)$-face, then for some choice of unit normals $\mathbf{n}_{F_i}$ to the facets*

$$s(F_1)\mathbf{n}_{F_1} + s(F_2)\mathbf{n}_{F_2} + s(F_3)\mathbf{n}_{F_3} + s(F_4)\mathbf{n}_{F_4} = 0 \tag{3.2}$$

If we have a collection S of faces of the tiling of arbitrary dimensions, for example, the whole tiling or the star of a face, by a canonical scaling of the collection S we mean a canonical scaling of the subset of S consisting of facets, ie. $(d-1)$-faces.

This definition was originally developed by Voronoi in [Vor09]. He used the term "canonically defined parallelotope" for what we call a "canonical scaling of the tiling". Our choice to use the word "scaling" is due to the fact that we are scaling (assigning lengths to) normal vectors to facets in the tiling.

We can interpret the scale factors in the canonical scaling of a tiling as the edge lengths of a *reciprocal graph*, a rectilinear graph in space whose vertices correspond to parallelotopes, and whose edges are orthogonal to their corresponding facets. See figure 2.2 on page 11 for an example of a reciprocal graph.

Two canonical scalings of face collections S and S' *agree* (on $S \cap S'$) if their restrictions to $S \cap S'$ are the same up to a common multiplier.

Consider a special case when $S = \mathrm{St}(F^{d-2})$.

(a) If F^{d-2} is hexagonal, then each two normal vectors to facets in its star are noncollinear, therefore a canonical scaling of S is unique up to a common multiplier.

This implies that if $\mathrm{St}(F^{d-2}) \subset S'$ and s' is a canonical scaling of S', then

$$\frac{s'(F_1)}{s'(F_2)} = \frac{s(F_1)}{s(F_2)} \tag{3.3}$$

(b) If F^{d-2} is quadruple, then there are two pairs of parallel facets in the star of F^{d-2}, say, F_1 and F_3, F_2 and F_4. Facets F_1 and F_2 are not parallel. For equation 3.2 to hold, we must $\mathbf{n}_{F_1} = -\mathbf{n}_{F_3}$, $\mathbf{n}_{F_2} = -\mathbf{n}_{F_4}$, and

$$\begin{aligned} s(F_1) = s(F_3) = a \\ s(F_2) = s(F_4) = b \end{aligned} \tag{3.4}$$

However, a and b can be any positive numbers. Therefore a canonical scaling of $\mathrm{St}(F^{d-2})$ is not unique up to a common multiplier.

Canonical scalings are important for our research because of the following theorem:

Theorem 5 *Consider a normal parallelotope tiling of $\mathbb{R}^d$. The following statements are equivalent:*

- *The tiling is affinely equivalent to the DV-tiling for some lattice.*
- *The tiling has a canonical scaling.*

Therefore, to prove the Voronoi conjecture for a tiling, it is sufficient to obtain a canonical scaling for the tiling.

We gave the proof of this result for the hexagonal tiling in section 2.3. Voronoi proved it for the case of primitive tilings. See paper [MD04] by Deza for a proof in the general case.

3.2 The quality translation theorem

To obtain canonical scalings, we use powerful topological ideas. Originally developed by Voronoi, they were generalized by Ryshkov and Rybnikov ([RKAR97]) into the Quality Translation Theorem.

The theorem is applicable to a wide class of polyhedral complexes. We need to review the complex-theoretic terminology to be able to present the theorem.

Definition 2 *([Lef42]) A polyhedral complex in $\mathbb{R}^n$ is a countable complex K with the following properties:*

1. *a p-dimensional element, or p-cell, c^p is a (bounded, convex) relatively open polytope in a p-dimensional affine space in $\mathbb{R}^n$;*
2. *the cells are disjoint;*
3. *the union of the cells $c' \prec c^p$ is the topological closure of c^p in $\mathbb{R}^d$;*
4. *If $\phi(c)$ is the union of the cells $c' \notin \mathrm{St}_K(c)$, then the topological closure of $\phi(c)$ does not intersect c.*

We denote by $|K|$ the support *of K, the union of all cells $c \in K$.*

We assume that the set $|K|$ has the topology induced from the Euclidean space. The *closure* $\mathrm{Cl}(c^p)$ of a cell c^p is the collection of cells c' with $c' \prec c^p$. The star $\mathrm{St}_K(c^p)$ of c^p is the collection of cells whose closures contain c^p. We write $\mathrm{St}(c^p)$ if it is clear which complex we are considering. The boundary of cell c^p is the closure of c^p without c^p.

The dimension n of a complex, $\dim(K)$, is the maximal dimension of an element. An upper index is used to indicate the dimension of the complex and its cells. A polyhedral complex K^n is called *dimensionally homogeneous* if each cell of dimension less than n is on the boundary of a cell of dimension n. A *strongly connected complex* is one where each two distinct n-cells can be connected by a sequence of n-cells where two consecutive cells intersect over a $(n-1)$-cell. The m-dimensional *skeleton* $\mathrm{Sk}_m(K^n)$ of the complex K^n is the complex consisting of cells of K^n of dimension m or less.

Examples of polyhedral complexes are polytopes, normal tilings of the Euclidean space, and their skeletons. In all of these examples, cells are the relative interiors of faces.

Definition 3 *([RKAR97]) An n-dimensional simply and strongly connected, dimensionally homogeneous polyhedral complex $K^n \subset \mathbb{R}^N$ is a QRR-complex*[1] *if all of the stars $\mathrm{St}(c^p)$, $p < n-1$ satisfy one of the following two local conditions.*

Local condition 1. *For $0 \leq p \leq n-3$, $\mathrm{St}(c^p)$ is combinatorially equivalent to the product of c^p and a relatively open cone with a simply connected and strongly connected $(n-p-1)$-dimensional finite polyhedron as its base.*

Local condition 2. *For $p = n-2$, $\mathrm{St}(c^p)$ is combinatorially equivalent to the product*

[1] *Authors of paper [RKAR97] use a notion of QRR-complex which is slightly more general.*

of c^p and a relatively open cone with a connected 1-dimensional finite polyhedron as its base. [2]

Proposition 1 *([RKAR97]) The following complexes are QRR:*

1. *Polytopes of arbitrary dimension.*

2. *Normal polytopal tilings of the Euclidean space $\mathbb{R}^d$.*

3. *Skeletons of dimension 2 or more of QRR-complexes.*

A *combinatorial path* on a QRR-complex K^n is a sequence of n-cells $[c_1^n, \ldots, c_k^n]$ where two consecutive cells either share a common $(n-1)$-cell, called a *joint*, or coincide. A *combinatorial circuit* is a path with at least two different n-cells, where the first and last cells coincide. A circuit is *k-primitive* if all its cells belong to the star of the same k-dimensional cell.

Suppose that we want to assign a positive number $s(c^n)$ to each n-cell of K^n. For example, we will need to calculate the scale factors in canonical scalings. Suppose that the numbers assigned to adjacent n-cells c_1^n, c_2^n must be related as follows:

$$\frac{s(c_2^n)}{s(c_1^n)} = T[c_1^n, c_2^n], \tag{3.5}$$

where values of $T[c_1^n, c_2^n]$ are given a priori. Compare this with equation 3.3 on page 20.

[2] *The term* cone *in the local conditions means the union of line segments joining a point outside the $(n-p-1)$-space spanned by the base polyhedron with each point in the base polyhedron, minus the base polyhedron itself.*

Assume $T[c_1^n, c_2^n]T[c_2^n, c_1^n] = 1$. A positive number can be assigned to combinatorial paths $[c_1^n, \ldots, c_k^n]$ by

$$T[c_1^n, \ldots, c_k^n] = T[c_1^n, c_2^n]T[c_2^n, c_3^n] \ldots T[c_{k-1}^n, c_k^n]. \tag{3.6}$$

The number $T[c_1^n, \ldots, c_k^n]$ is called the *gain* along the path $[c_1^n, \ldots, c_k^n]$, and the function T is called the *gain function.*

We can assign a number $s(c_0^n) = s_0$ to a fixed cell c_0^n and then try to use the gain function to assign a number to any other cell c^n. We connect c^n with c_0^n by a combinatorial path $[c_0^n, c_1^n, \ldots, c_k^n]$ where $c_k^n = c^n$ and then let

$$s(c^n) = s(c_0^n)T[c_0^n, c_1^n, \ldots, c_k^n]. \tag{3.7}$$

The question of consistency is whether *different* paths will lead to the same value of $s(c^n)$. Clearly that will be the case if and only if the gain along every combinatorial circuit is 1.

Theorem 6 *(Ryshkov, Rybnikov, [RKAR97]) (The quality translation theorem) The gain T along all combinatorial circuits is* 1 *if and only if the gain along all* $(n-2)$-*primitive combinatorial circuits is* 1.

In other words, if the process of assigning numbers to cells works consistently on the star of each $(n-2)$-face, then it works consistently on the whole complex.

The quality translation theorem can be used verbatim for assigning elements of a linear space $\mathbb{R}^d$ to cells in a complex. The gain function in this case is $\mathbb{R}^d$-valued. Authors of paper [RKAR97] formulated the theorem in an even more general setting which allows, for example, to assign colors to polytopes in a tiling.

3.3 Voronoi's result

To illustrate the use of theorem 6, we will prove the following theorem of Voronoi [Vor09]:

Theorem 7 *A primitive parallelotope tiling of d-dimensional space is affinely equivalent to a DV-tiling.*

Proof. For $d = 1$, the result is trivial. For $d = 2$, the only primitive parallelotope tiling is the hexagonal one. We've worked it out in the introduction to the thesis. We assume below that $d \geq 3$.

By theorem 5 on page 20, it is sufficient to obtain a canonical scaling of the tiling. This will be the strategy of the proof.

First we describe the structure of the star of a vertex v. By the assumption of the theorem, there are exactly $d+1$ parallelotopes $P_1, \dots, P_{d+1}$ in the star of v. Each edge incident with v has at least d parallelotopes in its star, however it cannot have all $d+1$ parallelotopes, because their intersection is v. Therefore each edge has exactly d parallelotopes in its star.

Since the collection of parallelotopes in the star of an edge uniquely defines the edge, there are at most $d+1$ edges incident with v (which is the number of ways to choose d parallelotopes out of $d+1$). On the other hand, there cannot be d or less edges because in this case all edges must be in the same parallelotope.

We have proved that there are exactly $d+1$ edges of the tiling which are incident with v. Each of the $d+1$ parallelotopes contains exactly d of these edges. Therefore v is a simple vertex of each parallelotope, and *each k edges incident with v are the edges of a face of the tiling of dimension k, for $k = 1, \dots, d$.* Since the vertex v was chosen arbitrarily, it follows that each $(d-k)$-dimensional face of the tiling is on the boundary of exactly $k+1$ parallelotopes, $k = 1, \ldots, d$. In particular, each $(d-2)$-face is hexagonal, since it has 3 parallelotopes in its star.

We denote the edge vectors of the $d+1$ edges by $e_1,\ldots,e_{d+1}$ so that the edges can be written as $[v, v+e_1]$, $[v, v+e_2],\ldots,$ $[v, v+e_{d+1}]$. Choosing the numbering appropriately, we have the vertex cone of parallelotope P_j at v equal to

$$C_j = v + \mathrm{cone}\{e_i : i \neq j\} \tag{3.8}$$

By cone(A) we mean the set of all linear combinations of vectors from A with nonnegative coefficients. Cones $C_1, \ldots, C_{d+1}$ form a tiling of the space (because in a small neighborhood of v they coincide with the parallelotopes).

We now prove that there is a canonical scaling of $\mathrm{St}(v)$. Consider the following polyhedron:

$$\Delta = \{x : x \cdot e_i \leq 1, i = 1, \ldots, d+1\}. \tag{3.9}$$

We have $\mathrm{int}(\Delta) \neq \emptyset$, for $0 \in \mathrm{int}(\Delta)$. The polyhedron Δ is bounded. For, assuming it is not bounded, there is a half-line $\{ut : t \geq 0\} \subset \Delta$, $u \in \mathbb{R}^d$, $u \neq 0$. We then have $e_i \cdot u \leq 0$ for all $i = 1, \ldots, d+1$ which is a contradiction of the fact that the cones C_j cover the space.

Therefore Δ is a simplex with vertices v_i given by $v_i \cdot e_j = 1$ for $i \neq j$, $v_i \cdot e_i < 1$.

We construct the canonical scaling to $\mathrm{St}(v)$ using simplex Δ. The affine hull of each facet $F_{kl} = P_k \cap P_l$ in the star can be written as $v + \mathrm{lin}\{e_i : i \neq k, l\}$. The edge vector $v_k - v_l$ of the simplex Δ is therefore orthogonal to F_{kl}. We let

$$s(F_{kl}) = |v_k - v_l|. \tag{3.10}$$

Let $\mathbf{n}_{kl} = \frac{v_k - v_l}{|v_k - v_l|}$. Then $\mathbf{n}_{kl}$ is a unit normal vector to F_{kl}. To prove that the scale factors we have assigned indeed form a canonical scaling, we need to check conformance to definition 1 on page 19. That is, we need to establish the following equation:

$$s(F_{kl})\mathbf{n}_{kl} + s(F_{lm})\mathbf{n}_{lm} + s(F_{mk})\mathbf{n}_{mk} = 0 \tag{3.11}$$

for all distinct $k, l, m \in 1 \ldots d+1$. This equation obviously holds; we just plug in the definitions of the scale factors and unit normals to check it.

We now construct a canonical scaling of the tiling. We have proved that each $(d-2)$-face F^{d-2} of the tiling is hexagonal. Let the facets in its star be F, G and H, and let $\alpha = \alpha_{F^{d-2}}$ be its canonical scaling. We apply the quality translation theorem (theorem 6 on page 24) to the $(d-1)$-skeleton of the tiling with the following gain function:

$$T[F, G] = \frac{\alpha(G)}{\alpha(F)}. \tag{3.12}$$

To apply the theorem, we need to check that the gain along an arbitrary $(d-3)$-primitive circuit $[F_1, \ldots, F_n, F_{n+1}]$, where $F_{n+1} = F_1$, is equal to 1. Suppose that the circuit is in the star of face F^{d-3} of the tiling. We have

$$T[F_1, \ldots, F_n, F_1] = \frac{\alpha_1(F_2)}{\alpha_1(F_1)} \frac{\alpha_2(F_3)}{\alpha_2(F_2)} \cdot \cdots \cdot \frac{\alpha_n(F_1)}{\alpha_n(F_n)} \tag{3.13}$$

where α_i is the canonical scaling of the star of $(d-2)$-face F_i^{d-2} joining F_i and F_{i+1}. We have $\mathrm{St}(F^{d-3}) \subset \mathrm{St}(v)$, where v is any vertex of F^{d-3}. Therefore $\mathrm{St}(F^{d-3})$ has a canonical scaling s, which can be obtained by restricting the canonical scaling to $\mathrm{St}(v)$.

Since $\mathrm{St}(F_i^{d-2}) \subset \mathrm{St}(F^{d-3})$, we have by equation 3.3 on page 20

$$\frac{\alpha_i(F_{i+1})}{\alpha_i(F_i)} = \frac{s(F_{i+1})}{s(F_i)}, i = 1, \ldots, n. \tag{3.14}$$

Factors in the numerator and denominator in equation 3.13 cancel each other and we have $T[F_1, \ldots, F_n, F_1] = 1$. By the quality translation theorem, the tiling has a canonical scaling, and therefore it is affinely equivalent to a DV-tiling, by theorem 5 on page 20. This proves the Voronoi conjecture for primitive tilings. $\square$

Chapter 4

Dual cells

In the previous chapter, we introduced canonical scalings of a tiling, whose existence is equivalent to the Voronoi conjecture. Our tool for constructing a canonical scaling is the quality translation theorem (theorem 6 on page 24). Applying the theorem to the $(d-1)$-skeleton of the tiling requires an understanding of $(d-3)$-primitive combinatorial circuits, or circuits of facets in the stars of $(d-3)$-faces.

In this chapter, we introduce *dual cells*, objects that are very convenient for describing stars of faces in the tiling. We give the classification of stars of faces of dimension $(d-2)$ and $(d-3)$.

4.1 Definition of dual cells

A DV-tiling can be equipped with a metrically dual tiling, known as a *Delaunay tiling*, or *L-tiling*. Vertices of the dual tiling are the centers of parallelotopes; the faces are orthogonal to the corresponding faces of the tiling[1].

[1] See Delaunay's paper [Del34] for details on the L-tiling.

However, we do not know whether an arbitrary parallelotope is a DV-domain or is affinely equivalent to one. In fact, proving that for 3-irreducible parallelotope tilings is the objective of our work. The following definition works whether or not we have a DV-tiling.

Definition 4 *Let* $0 \le k \le d$. *Given a face* F^{d-k} *of the tiling, the* dual cell D^k *corresponding to* F^{d-k} *is the convex hull of centers of parallelotopes in the star of* F^{d-k}.

It is so far an open problem to show that $\dim(D^k) + \dim(F^{d-k}) = d$ (or, equivalently, that $\dim(D^k) = k$). Therefore the number k will be called the *combinatorial dimension* of D^k. A *dual k-cell* is a dual cell of combinatorial dimension k. Dual 1-cells are called *dual edges*.

Suppose that D_1 and D_2 are dual cells. Then D_1 is called a *subcell* of D_2 if $\mathrm{Vert}(D_1) \subset \mathrm{Vert}(D_2)$. We denote this relationship by the symbol "$\prec$". It is important to distinguish between subcells and faces of D_2 in the polytope-theoretic sense. We do not know so far if they are the same thing.

A dual cell is called *asymmetric* if it does not have a center of symmetry.

We now prove some essential properties of dual cells.

1. The set of centers of parallelotopes in the star of face F^{d-k} *is in a convex position.* We use the notation $P(x)$ for the translate of the parallelotope of the tiling whose center is x, so that $2x - P(x) = P(x)$. Let v be an arbitrary vertex of F^{d-k}. Consider the polytope $Q = P(v)$ (note that Q does not belong to the tiling). The centers c of parallelotopes $P \in \mathrm{St}(F^{d-k})$ are vertices of Q. Indeed, v is a vertex of $P(c)$, therefore c is a vertex of $P(v)$. The claim follows from the fact that the set of vertices of Q is in a convex position.

2. The correspondence between faces of the tiling and dual cells is $1-1$. Indeed, if D is the dual cell corresponding to a face F of the tiling, then

$$F = \bigcap_{v \in \mathrm{Vert}(D)} P(v). \tag{4.1}$$

This result follows from the fact that a proper face of a polytope is the intersection of facets of the polytope which contain it, which in turn is a partial case of theorem 9 on page 54 of [MS71].

3. If D_1, D_2 *are dual cells corresponding to faces* F_1, F_2, *then* $\mathrm{Vert}(D_1) \subset \mathrm{Vert}(D_2)$ *is equivalent to* $F_2 \prec F_1$. This result follows from the definition of the dual cell and the previous formula.

Results 2 and 3 can be reformulated as follows:

4. Dual cells form a dual abstract complex to the tiling.

One corollary is that if D_1 and D_2 are dual cells and their vertex sets intersect, then $\mathrm{conv}(\mathrm{Vert}(D_1) \cap \mathrm{Vert}(D_2))$ is a dual cell.

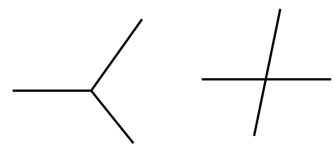
4.2 Dual 2, 3-cells

In this section, we classify all dual dual 2- and 3-cells. First we quote the classification of fans of $(d-2)$- and $(d-3)$-dimensional faces.

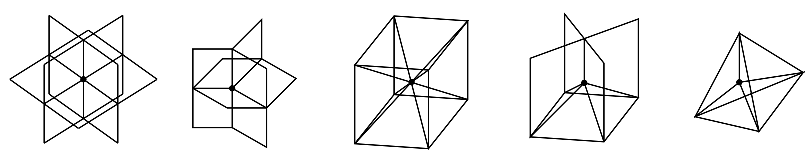
The fan of a face is defined as follows. For each parallelotope P in the star of a face F^{d-k}, take the cone $\mathcal{C}_P$ bounded by the facet supporting inequalities for facets of P which contain F^{d-k}. The cones $\mathcal{C}_P$ form a face-to-face partitioning of the d-dimensional space $\mathbb{R}^d$. All facets in the partitioning are parallel to the face F^{d-k}. By taking a section of the partitioning by a k-dimensional plane orthogonal to F^{d-k}, we get the *fan* of the tiling at face F^{d-k}.

Delaunay in [Del29] proves the following lemma.

Lemma 2 *The fans of parallelotopes of any dimension d at a face of dimension $(d-2)$ can only be of the following two combinatorial types:*



The fans of parallelotopes of any dimension d at a face of dimension $(d-3)$ can only be of the following 5 combinatorial types:



I II III IV V

A fan of type (I) is simply the partitioning of 3-space by 3 linearly independent planes. A type (II) fan has 3 half-planes meeting at a line, together with a plane which intersects the line. It is the fan at a vertex in the tiling of space by hexagonal prisms. A fan of type (III) has 6 cones with the vertex in the center of a parallelepiped; the cones are generated by the 6 facets of the parallelepiped. A fan of type (IV) has 5 cones. A fan of type (V) has the minimal number of 4 cones.

Lemma 3 *The stars of $(d-2)$- and $(d-3)$-faces have canonical scalings. Dual cells D^k of combinatorial dimension $k = 2$ and $k = 3$, corresponding to faces F^{d-k} of the tiling, are given by the following table. Nonempty faces of D^k are dual cells corresponding to faces in the star of face F^{d-k}.*

k	Fan of F^{d-k}	D^k	Canonical scaling of $\mathrm{St}(F^{d-k})$ unique up to a common multiplier	Tiling locally irreducible at F^{d-k}
2	(a)	triangle	Yes	Yes
	(b)	parallelogram	No	No
3	I	parallelepiped	No	No
	II	triangular prism	No	No
	III	octahedron	Yes	Yes
	IV	pyramid over parallelogram	Yes	Yes
	V	simplex	Yes	Yes

(4.2)

Table 4.1: Dual 2- and 3-cells

Proof. Results of this lemma were established in Delaunay's paper [Del29]. ∎

From this lemma, it follows that a tiling is 3-irreducible if and only if all dual 3-cells are simplices, octahedra or pyramids.

Chapter 5

Coherent parallelogram cells

We now expose the obstacle on the way to constructing a canonical scaling for the tiling: the *incoherent parallelogram dual cells.*

The most important result is theorem 9 on page 38, which implies that incoherent parallelogram cells come in groups of at least 5 (if they come at all). This helps to prove their nonexistence.

5.1 Coherent parallelogram cells

Definition 5 *Consider a parallelotope tiling of d-dimensional space, where* $d \geq 4$*. Let* Π *be a parallelogram dual cell,* D^4 *a dual* 4*-cell,* D_1^3*,* D_2^3 *pyramid dual cells with* $\Pi \prec D_1^3 \prec D^4$*,* $\Pi \prec D_2^3 \prec D^4$*. Let* F^{d-2}*,* F^{d-4} F_1^{d-3}*,* F_2^{d-3} *be the corresponding faces of the tiling.*

The parallelogram dual cell Π *is called* coherent *with respect to* D^4 *if the canonical scalings of* $\mathrm{St}(F_1^{d-3})$*,* $\mathrm{St}(F_2^{d-3})$ *agree on* $\mathrm{St}(F^{d-2})$*. The face* F^{d-2} *is called coherent with respect to* F^{d-4} *if* Π *is coherent with respect to* D^4*.*

Note that canonical scalings of $\mathrm{St}(F_1^{d-3})$ and $\mathrm{St}(F_2^{d-3})$ in the definition are unique because the corresponding dual cells D_1^3, D_2^3 are pyramids, see lemma 3 on page 32.

Information on canonical scalings can be found on page 19.

The "rhombus" diagram of the faces below shows the relationships between faces and dual cells in the definition (arrows indicate "$\prec$" relationships).

Note that faces F_1^{d-3} and F_2^{d-3} are uniquely defined by F^{d-4} and F^{d-2}, since F^{d-4} is a codimension 2 face of F^{d-2}. Therefore dual cells D_1^3, D_2^3 are uniquely defined by D^4 and Π.

$$
\begin{array}{ccccc}
 & F^{d-4} & \\
\swarrow & & \searrow \\
F_1^{d-3} & & F_2^{d-3} \\
\searrow & & \swarrow \\
 & F^{d-2} &
\end{array}
\qquad
\begin{array}{ccccc}
 & D^4 & \\
\nearrow & & \nwarrow \\
D_1^3 & & D_2^3 \\
\nwarrow & & \nearrow \\
 & \Pi &
\end{array}
\tag{5.1}
$$

Theorem 8 *Consider a* 3*-irreducible tiling by parallelotopes. The tiling has a canonical scaling if and only if all parallelogram dual cells* Π *are coherent with respect to all* D^4 *with* $\Pi \subset D^4$.

Proof. Note that saying: parallelogram dual cell $\Pi \prec D^4$ is coherent with respect to D^4 is equivalent to saying that F^{d-2}, a face of the tiling corresponding to Π, is coherent to F^{d-4}, a face of the tiling corresponding to D^4.

The necessity part is easy. Consider faces of the tiling in diagram 5.1. If the tiling has a canonical scaling s, then canonical scalings of $\mathrm{St}(F_1^{d-3})$ and $\mathrm{St}(F_2^{d-3})$ can be obtained by restricting s:

$$s|_{\mathrm{St}(F_1^{d-3})}$$
$$s|_{\mathrm{St}(F_2^{d-3})} \tag{5.2}$$

If we in turn restrict these to $\mathrm{St}(F^{d-2}) = \mathrm{St}(F_1^{d-3}) \cap \mathrm{St}(F_2^{d-3})$, we get the same canonical scaling of $\mathrm{St}(F^{d-2})$, namely,

$$s|_{\mathrm{St}(F^{d-2})} \tag{5.3}$$

therefore the canonical scalings in (5.2) agree on $\mathrm{St}(F^{d-2})$ and F^{d-2} is coherent with respect to F^{d-4}. Since the faces F^{d-2} and F^{d-4} can be chosen arbitrarily, the necessity part of the theorem is proved.

Now we prove the sufficiency. We begin with a result for quadruple face that we will need in the proof. Take a quadruple face F^{d-2} of the tiling and consider two arbitrary $(d-3)$-faces $F^{d-3}, G^{d-3} \prec F^{d-2}$. We observe that the canonical scalings to $\mathrm{St}(F^{d-3})$ and $\mathrm{St}(G^{d-3})$ agree on $\mathrm{St}(F^{d-2})$. Indeed, we can connect two $(d-3)$-faces F^{d-3}, G^{d-3} on the boundary of F^{d-2} by a combinatorial path on the same boundary:

$$F^{d-3} = F_1^{d-3}, \ldots, F_n^{d-3} = G^{d-3} \tag{5.4}$$

where two consecutive $(d-3)$-faces F_i^{d-3} and F_{i+1}^{d-3} share a $(d-4)$-face F_i^{d-4}. From the conditions of the theorem, it follows that all dual cells corresponding to F_i^{d-3}, $i = 1, \ldots, n$ are pyramids and therefore the canonical scaling to $\mathrm{St}(F_i^{d-3})$ is unique up to a common multiplier.

From the condition of the theorem, F^{d-2} is coherent with respect to F_i^{d-4} for all

$i = 1, \ldots, n-1$, which means that the canonical scaling of $\mathrm{St}(F_i^{d-3})$ agrees with the canonical scaling of $\mathrm{St}(F_{i+1}^{d-3})$. By the chain argument, we see that the canonical scalings of $\mathrm{St}(F^{d-3}) = \mathrm{St}(F_1^{d-3})$ and $\mathrm{St}(G^{d-3}) = \mathrm{St}(F_n^{d-3})$ agree on $\mathrm{St}(F^{d-2})$.

Now we use the quality translation theorem to obtain the canonical scaling of the tiling. Let F^{d-2} be a $(d-2)$-face. It can be either quadruple or hexagonal. We define the gain function as follows: suppose that facets F_1, F_2 of the tiling are in the star of the $(d-2)$-face F^{d-2}. We set

$$T[F_1, F_2] = \frac{\alpha(F_2)}{\alpha(F_1)}, \tag{5.5}$$

where $\alpha = \alpha_{F^{d-3}}$ is the canonical scaling of $\mathrm{St}(F^{d-3})$ for an arbitrary face F^{d-3} with $F^{d-3} \prec F^{d-2}$, so that $\mathrm{St}(F^{d-2}) \subset \mathrm{St}(F^{d-3})$.

The definition of $T[F_1, F_2]$ is consistent, that is, the value of $\frac{\alpha(F_2)}{\alpha(F_1)}$ does not depend on the choice of F^{d-3} if F^{d-2} is hexagonal. Indeed, if face F^{d-2} is hexagonal, then the canonical scaling to $\mathrm{St}(F^{d-2})$ is unique up to a common multiplier. If $\beta = \beta_{F^{d-2}}$ is some fixed canonical scaling to $\mathrm{St}(F^{d-2})$, then the restriction of α to $\mathrm{St}(F^{d-2})$ must agree with it:

$$\frac{\alpha(F_2)}{\alpha(F_1)} = \frac{\beta(F_2)}{\beta(F_1)}, \tag{5.6}$$

therefore the definition of $T[F_1, F_2]$ does not depend on the choice of F^{d-3}.

Suppose now that face F^{d-2} is quadruple. As we argued above, if we have another face $G^{d-3} \prec F^{d-2}$, then the canonical scalings $\alpha = \alpha_{F^{d-3}}$, $\alpha' = \alpha_{G^{d-3}}$ to $\mathrm{St}(F^{d-3})$ and $\mathrm{St}(G^{d-3})$ agree on $\mathrm{St}(F^{d-2})$, therefore

$$\frac{\alpha(F_2)}{\alpha(F_1)} = \frac{\alpha'(F_2)}{\alpha'(F_1)} \tag{5.7}$$

We have proved that the value of $T[F_1, F_2]$ is well-defined.

By construction, the gain function is equal to 1 on $(d-3)$-primitive circuits (the reader may find a detailed argument in the proof of theorem 7 on page 25). Application of the quality translation theorem gives an assignment s of positive numbers to facets of the tiling.

Equation (5.5) guarantees that the restrictions of s to stars of $(d-2)$-faces are canonical scalings. Therefore the conditions of definition 1 on page 19 are satisfied by s. We have proved that the tiling has a canonical scaling. $\square$

5.2 Sufficient conditions for coherency of a parallelogram dual cell

Theorem 9 *Consider a 3-irreducible normal parallelotope tiling. Let D^4 be a dual cell, $\Pi \subset D^4$ a parallelogram subcell. Suppose Π has a vertex v such that all parallelogram subcells $\Pi' \subset D^4$ with $v \in \Pi'$ are coherent except, perhaps, Π. Then Π is coherent. In particular, if Π is the only parallelogram in D^4 that contains v, then Π is coherent.*

Speaking informally, incoherent parallelograms "come in droves": each vertex of an incoherent parallelogram must also be a vertex of another incoherent parallelogram.

Proof. Let F^{d-4} be the face of the tiling corresponding to D^4, P the parallelotope of the tiling with center v, F^{d-2} the face corresponding to Π, F_1^{d-3}, F_2^{d-3} faces with the property that $F^{d-4} \prec F_1^{d-3}, F_2^{d-3} \prec F^{d-2}$. Let D_1^3, D_2^3 be the dual cells corresponding to faces F_1^{d-3}, F_2^{d-3}. By our assumption about the tiling, they are pyramids. The relationship of the introduced polytopes can be seen in diagram 5.12 on page 41, arrows indicate inclusion.

Let I_1, I_3 be the edges of Π containing v, and let I_2,I_4 be the edges of pyramids D_1^3, D_2^3 containing v and different from I_1,I_3. Let $F_1, \ldots, F_4$ be the facets of the tiling corresponding to dual edges $I_1, \ldots, I_4$. Take canonical scalings s_1 and s_2 to $\mathrm{St}(F_1^{d-3})$ and $\mathrm{St}(F_2^{d-3})$. To show that they agree on the star of F^{d-2}, it is sufficient to check that they assign proportional scale factors to facets F_1 and F_3, ie.,

$$\frac{s_1(F_1)}{s_1(F_3)} = \frac{s_2(F_1)}{s_2(F_3)} \tag{5.8}$$

To prove this result, we use the quality translation theorem (theorem 6 on page 24). Consider face collections K' and K:

$$\begin{aligned} K' &= \{F : F^{d-4} \prec F \prec P\}, \\ K &= K' \setminus \{F^{d-4}, F_1^{d-3}, F_2^{d-3}, F^{d-2}, P\}. \end{aligned} \tag{5.9}$$

We define the gain function on combinatorial paths in K. Consider a path $[G_1, G_2]$ on K, where facets G_1 and G_2 share a face $G^{d-2} \neq F^{d-2}$. Note that G^{d-2} is coherent with respect to F^{d-4}: since Π is the only parallelogram cell with $v \prec \Pi \prec D^4$ which may a priori be incoherent with respect to D^4, F^{d-2} is the only potentially incoherent $(d-2)$-face with $F^{d-4} \prec F^{d-2} \prec P$.

There are exactly two $(d-3)$ faces G_1^{d-3}, G_2^{d-3} with $F^{d-4} \prec G_i^{d-3} \prec G^{d-2}$. We define $T[G_1, G_2]$ by

$$T[G_1, G_2] = \frac{s(G_2)}{s(G_1)} = \frac{s'(G_2)}{s'(G_1)}. \tag{5.10}$$

where s, s' are the canonical scalings of $\mathrm{St}(G_1^{d-3})$ and $\mathrm{St}(G_2^{d-3})$. The two last fractions are equal because the canonical scalings s, s' agree on the star of G^{d-2}. If G^{d-2} is hexagonal, then this result follows from the uniqueness (up to a common multiplier) of a canonical scaling to $\mathrm{St}(G^{d-2})$. If G^{d-2} is quadruple, then it is coherent with respect to F^{d-4}, therefore s and s' agree on $\mathrm{St}(F^{d-2})$.

Note that equation (5.10) implies that the gain on $(d-3)$-primitive circuits is equal to 1 (see a detailed argument in the proof of theorem 7 on page 25).

The statement that the parallelogram Π is coherent (equation (5.8)) can be rewritten as:

$$T[F_1, F_2, F_3, F_4, F_1] = 1. \tag{5.11}$$

Indeed, $F_1, F_2, F_3 \in \mathrm{St}(F_1^{d-3})$ and $F_1, F_4, F_3 \in \mathrm{St}(F_2^{d-3})$, hence

$$\frac{s_1(F_3)}{s_1(F_1)} = \frac{s_1(F_3)}{s_1(F_2)}\frac{s_1(F_2)}{s_1(F_1)} = T[F_1, F_2, F_3]$$

and

$$\frac{s_2(F_1)}{s_2(F_3)} = \frac{s_2(F_1)}{s_1(F_4)}\frac{s_1(F_4)}{s_2(F_3)} = T[F_3, F_4, F_1].$$

By theorem 16 on page 71 of [MS71], $K' = \{F : F^{d-4} \prec F \prec P\}$ is combinatorially isomorphic to a 3-dimensional polytope Q (ie., the lattices of the two complexes are isomorphic). Denote the isomorphism by δ. $I = \delta(F^{d-2})$ is an edge of Q. All other edges of Q are the images of coherent $(d-2)$-faces. Facets of Q are the images of facets in K'. Under the isomorphism, complex K' is mapped onto the boundary of Q minus the edge I and its two endpoints.

Removing the edge from the boundary of Q leaves it simply and strongly connected, therefore the quality translation theorem (theorem 6 on page 24) can be applied to K'. It follows that the gain on the circuit $[F_1, F_2, F_3, F_4, F_1]$ is equal to 1, and the theorem's result follows[1]. $\square$

$$
\begin{tikzcd}
 & D^4 & \\
D_1^3 \arrow[ur] & & D_2^3 \arrow[ul] \\
 & \Pi \arrow[ul] \arrow[ur] & \\
I_1 \arrow[ur] & & I_3 \arrow[ul] \\
 & v \arrow[ul] \arrow[ur] &
\end{tikzcd}
\quad\Bigg|\quad
\begin{tikzcd}
 & F^{d-4} \arrow[dl] \arrow[dr] & \\
F_1^{d-3} \arrow[dr] & & F_2^{d-3} \arrow[dl] \\
 & F^{d-2} \arrow[dl] \arrow[dr] & \\
F_1 \arrow[dr] & & F_3 \arrow[dl] \\
 & P &
\end{tikzcd}
\quad\Bigg|\quad
\begin{tikzcd}
 & \emptyset \arrow[dl] \arrow[dr] & \\
v_1 \arrow[dr] & & v_2 \arrow[dl] \\
 & I \arrow[dl] \arrow[dr] & \\
\delta(F_1) \arrow[dr] & & \delta(F_3) \arrow[dl] \\
 & Q &
\end{tikzcd}
\tag{5.12}
$$

Theorem 10 *Let D^4 be a dual 4-cell in a parallelotope tiling. Suppose that D^4 is a centrally symmetric bipyramid over a parallelepiped C, and that subcells of D^4 coincide with its faces. Then all parallelogram dual subcells of D^4 are coherent with respect to D^4.*

Proof. Polytope D^4 is the convex hull of a parallelepiped C and two vertices v, v' outside the space spanned by C, whose midpoint is the center of C. The facets of D^4 are 12 pyramids over parallelograms.

All edges of D^4 are dual 1-cells. D^4 has 28 edges: 8 edges incident with v, 8 edges incident with v', and 12 edges of C. Let $\mathcal{F}$, $\mathcal{F}'$, and $\mathcal{M}$ be the collections of facets of the tiling, corresponding to these 3 groups of dual edges.

Consider the complex K, the union of (closed) pyramid facets of D^4 meeting at v. Complex K contains 6 pyramids over parallelograms. Each pyramid D is a dual cell. Let F^{d-3} be the corresponding face of the tiling. By lemma 3 (page 32), there

[1] Strictly speaking, the boundary of Q with an edge and its endpoints removed is not a QRR-complex, but we can cut away the edge together with a small neighborhood, producing a 2-dimensional polyhedral QRR-complex.

is a canonical scaling of $\mathrm{St}(F^{d-3})$, which we will denote s_D. We will fix s_D for each of the pyramids in K.

We now find a positive multiplier $m(D)$ for each pyramid D so that the 6 canonical scalings

$$m(D)s_D \tag{5.13}$$

together define a canonical scaling on the collection

$$\mathcal{F} \cup \mathcal{M}. \tag{5.14}$$

In other words, we require that whenever two pyramids D_1, D_2 share a triangular dual cell T, then for each facet F corresponding to one of the edges of T, we have

$$\begin{gathered} m(D_1)s_{D_1}(F) = m(D_2)s_{D_2}(F), \\ \text{or} \\ \frac{s_{D_1}(F)}{s_{D_2}(F)} = \frac{m(D_2)}{m(D_1)}. \end{gathered} \tag{5.15}$$

We define the gain function on K as follows:

$$T[D_1, D_2] = \frac{s_{D_1}(F)}{s_{D_2}(F)}. \tag{5.16}$$

The definition does not depend on the choice of F. This is easy to see. Let F_1^{d-3}, F_2^{d-3}, F^{d-2} be the faces of the tiling corresponding to dual cells D_1, D_2, T. Since T is a triangle, face F^{d-2} is hexagonal and the canonical scaling to $\mathrm{St}(F^{d-2})$ is unique up to a factor. Since s_{D_1} and s_{D_2} are canonical scalings to $\mathrm{St}(F_1^{d-3})$ and $\mathrm{St}(F_2^{d-3})$, and $\mathrm{St}(F^{d-2}) = \mathrm{St}(F_1^{d-3}) \cap \mathrm{St}(F_2^{d-3})$, for each two facets $F, F' \in \mathrm{St}(F^{d-2})$ we have (see equation (3.3) on page 20):

$$\frac{s_{D_1}(F)}{s_{D_2}(F)} = \frac{s_{D_1}(F')}{s_{D_2}(F')}. \tag{5.17}$$

We have proved that the definition of $T[D_1, D_2]$ does not depend on the choice of $F \in \mathrm{St}(F^{d-2})$.

The dimension of complex K is 3. To apply the quality translation theorem (theorem 6 on page 24), we need to check that a 1-primitive circuit $[D_1, \ldots, D_n, D_1]$ has gain 1. Indeed, since a 1-primitive circuit in K is in the star of an edge, the facet F corresponding to that edge belongs to the stars of all faces F_i^{d-3} corresponding to D_i, and the gain function can be calculated using the value of $s_{D_i}(F)$ alone as

$$T[D_1, \ldots, D_n, D_1] = \frac{s_{D_1}(F)}{s_{D_2}(F)} \frac{s_{D_2}(F)}{s_{D_3}(F)} \cdot \ldots \cdot \frac{s_{D_n}(F)}{s_{D_1}(F)} = 1. \tag{5.18}$$

Clearly K is a QRR-complex. Therefore, by the quality translation theorem, we obtain scale factors $m(D)$ with the required property.

We now have canonical scalings $m(D)s_D$ which together define a common canonical scaling s on $\mathcal{F} \cup \mathcal{M}$.

Each 4 facets of the tiling corresponding to a group of parallel edges of C receive the same scale factor s. To prove this, it is sufficient to check that parallel facets F_1, F_2 corresponding to parallel edges of a parallelogram cell get the same scale factor. Let $D \in K$ be a pyramid which contains the parallelogram, and let F^{d-3} be the corresponding face of the tiling. We have

$$\frac{s(F_1)}{s(F_2)} = \frac{s_D(F_1)}{s_D(F_2)} = 1, \tag{5.19}$$

because s_D is a canonical scaling to $\mathrm{St}(F^{d-3})$.

Note that D^4 is centrally symmetric, as is the star $\mathrm{St}(F^{d-4})$. The image of facet collection $\mathcal{F} \cup \mathcal{M}$ under the central symmetry is $\mathcal{F}' \cup \mathcal{M}$.

We have proved in the last paragraph that parallel facets in $\mathcal{M}$ have the same scale factors s. This means that s can be continued to the whole set

$$\mathcal{F} \cup \mathcal{F}' \cup \mathcal{M} \tag{5.20}$$

by the central symmetry $*$ of D^4, by assigning $s(F) = s(*(F))$ for $F \in \mathcal{F}' \cup \mathcal{M}$. It is easy to see that the continuation of s is a canonical scaling of $\mathrm{St}(F^{d-4})$.

It follows that all parallelogram subcells of D^4 are coherent. This completes the proof of the theorem. $\square$

Chapter 6

Geometric results on dual cells

We defined dual cells in chapter 4 and used the dual cell terminology in the previous chapter to state some combinatorial properties of incoherent parallelogram cells.

It turns out that dual cells carry not only combinatorial, but also geometric information about the parallelotope tiling. In this chapter, we prove some geometric, or polytope-theoretic properties of dual cells. The results are applicable to any normal tiling by parallelotopes and we believe they may be useful for further research in parallelotope theory.

The main result of this chapter deals with dual cells that are affinely equivalent to cubes of arbitrary dimension. In particular, we prove that a pair of parallelogram subcells in an arbitrary dual 4-cell is either

- complementary (parallelograms share a vertex and span a 4-space),
- adjacent (parallelograms share an edge and span a 3-space),
- translate (parallelograms are translated copies of each other and span a 3-space), or

- skew (parallelograms have exactly one common edge direction and span a 4-space).

This result (corollary 6 on page 71) follows from the more general theorem 11 on page 63.

Throughout the chapter, we consider a normal tiling of the space $\mathbb{R}^d$ by parallelotopes, and denote by Λ the lattice formed by the centers of parallelotopes (known as the *lattice of the tiling*). We assume that one of the parallelotopes of the tiling is centered at 0.

6.1 A series of lemmas on dual cells

Lemma 4 *Let D^k be the dual cell corresponding to a face F^{d-k} of the tiling, $0 < k \leq d$, $d \geq 1$. Suppose that $t \in \Lambda$, $t \neq 0$, and that h is the projection along F^{d-k} onto a complementary affine space L. Then:*

1. $|h(\operatorname{Vert}(D^k))| = |\operatorname{Vert}(D^k)|$.

2. *If $D^k \cap (D^k + t) \neq \emptyset$, then $D = D^k \cap (D^k + t)$ is a dual cell and a face of both D^k and $D^k + t$. There is a hyperplane N in $\mathbb{R}^d$ which separates*[1] *D^k and $D^k + t$, with*

$$\begin{aligned} D = N \cap D^k = N \cap (D^k + t), \\ \operatorname{lin}(F^{d-k} - F^{d-k}) \subset N - N \end{aligned} \tag{6.1}$$

3. *The same hyperplane N also separates $h(D^k)$ and $h(D^k + t)$.*

[1] *Hyperplane N separates two convex sets A and B if* $\operatorname{relint}(A)$ *and* $\operatorname{relint}(B)$ *belong to different open halfspaces bounded by H.*

4. $\mathrm{Vert}(h(D^k)) = h(\mathrm{Vert}(D^k))$.

Proof. For $x \in \mathbb{R}^d$, $P(x)$ will denote the translated copy of the parallelotope P so that x is the center of $P(x)$.

As a preparation for the proof, we find a way to *inscribe an inverted (centrally symmetric) copy of the dual cell into the parallelotope.* Fix a parallelotope $P \in \mathrm{St}(F^{d-k})$. We introduce an equivalence relation $\sim$ on faces of the tiling. Two faces F_1, F_2 are equivalent if there is a vector $v \in \Lambda$ so that $F_2 = F_1 + v$. Consider the equivalence class which contains face F^{d-k}. Fix a point $x \in \mathrm{relint}(F^{d-k})$ and for every equivalent face $F = F^{d-k} + v$ assign the point $x(F) = x + v$ to F. If F^{d-k} is centrally symmetric, then we choose x to be the center of symmetry of F^{d-k}; we will need this further in the thesis.

Let c be the center of symmetry of P. The polytope

$$S = \mathrm{conv}\{x(F) : F \sim F^{d-k}, F \prec P\}$$

is the image of D^k under the central symmetry $*$ with center $\frac{x+c}{2}$. The transformation $*$ is defined as follows:

$$*(y) = x + c - y. \tag{6.2}$$

Indeed, for each parallelotope $Q \in \mathrm{St}(F^{d-k})$ with center c_1, $F^{d-k} + (c - c_1)$ is a face of P, and $x(F^{d-k} + (c - c_1)) = x + (c - c_1) = *(c_1)$. Conversely, if $F = F^{d-k} + v$ is a face of P, then the parallelotope with center $c - v$ belongs to $\mathrm{St}(F^{d-k})$ so $c - v \in \mathrm{Vert}(D^k)$. Note that $c - v = *(x + v) = *(x(F))$. Therefore sets $\{x(F) : F \sim F^{d-k}, F \prec P\}$ and $\{y : P(y) \in \mathrm{St}(F^{d-k})\}$ are mapped onto each other by $*$. Since S and D^k are their convex hulls, we have

$$*(S) = D^k \tag{6.3}$$

Since all vertices of S are contained in P,

$$S \subset P \tag{6.4}$$

1. First we prove that $|h(\mathrm{Vert}(D^k))| = |\mathrm{Vert}(D^k)|$. If this is not true, then there are two distinct parallelotopes P, P' in the star of F^{d-k} such that $P' = P + \lambda$ and $\lambda \in \mathrm{lin}(F^{d-k} - F^{d-k})$. Since F^{d-k} is a face of $P + \lambda$, $F^{d-k} - \lambda$ is a face of P. This is only possible if $\lambda = 0$, so $P = P'$.

2. Now we prove the second statement of the lemma. Suppose that $D^k \cap (D^k + t) \neq \emptyset$, $t \in \Lambda$, $t \neq 0$.

We have $S \cap (S - t) \neq \emptyset$, so $B = P \cap (P - t) \neq \emptyset$. Take a hyperplane H supporting the polytope P so that $H \cap P = B$.

Let m be the midpoint between the centers of P and $P - t$. Remember that the parallelotopes of the tiling are centrally symmetric. It follows that the central symmetry at m maps P to $P - t$, and vice versa, so m is the center of symmetry of B. Therefore m belongs to H. Applying the central symmetry at m, we see that H supports $P - t$ as well, and $H \cap (P - t) = B$. The hyperplane H separates polytopes P and $P - t$ (that is, they are contained in different half-spaces bounded by H).

We prove that $S \cap (S - t)$ is a face of both of the polytopes S and $S - t$. Let H^+, H^- be the open half-spaces bounded by H, whose closures contain P and $P - t$

respectively. We decompose the set of vertices of polytopes S and $S-t$ as follows:

$$
\begin{aligned}
\mathrm{Vert}(S) \cup \mathrm{Vert}(S-t) = \{x(F) : F \sim F^{d-k}, F \prec P \text{ or } F \prec (P-t)\} \\
= \\
\{x(F) : F \sim F^{d-k}, F \prec P, F \not\prec (P-t)\} \cup \\
\{x(F) : F \sim F^{d-k}, F \not\prec P, F \prec (P-t)\} \cup \\
\{x(F) : F \sim F^{d-k}, F \prec P \text{ and } F \prec (P-t)\} \\
= \\
\{x(F) : F \sim F^{d-k}, F \prec P, F \not\prec B\} \cup \\
\{x(F) : F \sim F^{d-k}, F \prec (P-t), F \not\prec B\} \cup \\
\{x(F) : F \sim F^{d-k}, F \prec B\} \\
= \\
A^+ \cup A^- \cup M.
\end{aligned}
\tag{6.5}
$$

Note that for faces $F \sim F^{d-k}$ of polytopes P and $P-t$, the inclusion $x(F) \in H$ is equivalent to $F \prec B$. Therefore set A^+ is contained in H^+, set A^- is contained in H^-, set M is contained in H. Neither of the sets A^+, A^-, M is empty. Indeed, if M is empty, then $S \cap (S-t) = \emptyset$; if, say, A^+ is empty, then A^- is also empty and all faces of P equivalent to F^{d-k} are contained in B, which is impossible (if $F \prec B$, you can always take a facet containing B and take $F - u \prec P$, where u is the facet vector).

We have $\mathrm{Vert}(S) = A^+ \cup M$ and $\mathrm{Vert}(S-t) = A^- \cup M$. It follows that $\mathrm{conv}(M)$ is a face of both of the polytopes S and $S-t$, $\mathrm{conv}(M) = S \cap (S-t)$. Also H supports S and $S-t$ and separates them.

The condition

$$\operatorname{lin}(F^{d-k} - F^{d-k}) \subset H - H \tag{6.6}$$

holds because the hyperplane H contains a face $F \sim F^{d-k}$.

Let $N = *(H)$. Applying the central symmetry $*$ to polytopes S and $S - t$, we get D^k and $D^k + t$. Statement 2 of the lemma follows from the results we have proved for S and $S - t$.

3. To prove statement 3, note that hyperplane $*(H)$ is parallel to F^{d-k}, therefore projection h maps the halfspaces $*(H^+)$ and $*(H^-)$ into themselves. Since $\operatorname{relint}(D^k)$ and $\operatorname{relint}(D^k + t)$ belong to different open halfspaces $*(H^+)$ and $*(H^-)$, sets $\operatorname{relint}(h(D^k)) = h(\operatorname{relint}(D^k))$ and $\operatorname{relint}(h(D^k + t)) = h(\operatorname{relint}(D^k + t))$ are contained in the same two open halfspaces.

4. Finally, we prove that $\operatorname{Vert}(h(D^k)) = h(\operatorname{Vert}(D^k))$. Each vertex v of S belongs to the relative interior of a face of P which is a translate of F^{d-k}, therefore $h(v)$ is a vertex of $h(P)$. This means that for each vertex v of D^k, $h(v)$ is a vertex of $h(*(P))$. It follows that $h(v)$ is a vertex of $h(D^k)$. Therefore $h(\operatorname{Vert}(D^k)) \subset \operatorname{Vert}(h(D^k))$. This, together with statement 1 of the lemma, establishes statement 4 and completes the proof of the lemma. □

Corollary 3 *Let D be a dual cell. Then* $\Lambda \cap D = \operatorname{Vert}(D)$.

Proof. The inclusion $\operatorname{Vert}(D) \subset \Lambda \cap D$ is trivial. We need to prove $\Lambda \cap D \subset \operatorname{Vert}(D)$.

Take $v \in \Lambda \cap D$. Let y be any vertex of D. Dual cells D and $D + (v - y)$ intersect, v is a vertex of $D + (v - y)$, and it belongs to the intersection $D \cap D + (v - y)$, which is a face of polytopes D and $D + (v - y)$ by statement 2 of lemma 4. Therefore v is a vertex of D. □

Corollary 4 *Let D_1, D_2 be dual cells. Then D_1 is a subcell of D_2 if and only if $D_1 \subset D_2$.*

Proof. By definition D_1 is a subcell of D_2 if and only if $\text{Vert}(D_1) \subset \text{Vert}(D_2)$. So, if that holds, then $D_1 \subset D_2$.

Conversely, suppose that $D_1 \subset D_2$. We need to prove that $\text{Vert}(D_1) \subset \text{Vert}(D_2)$. We have $\text{Vert}(D_1) \subset \Lambda \cap D_2$. Since by the previous corollary $\Lambda \cap D_2 = \text{Vert}(D_2)$, each vertex of D_1 is also a vertex of D_2. $\square$

Corollary 5 *Let D be a dual cell. Each parity class in the lattice Λ of the tiling (a class modulo 2Λ) is represented at most once among vertices of D.*

Proof. Suppose that λ_1 and λ_2 are distinct vertices of D, and $\lambda_1 - \lambda_2 \in 2\Lambda$. Then $\lambda = \frac{\lambda_1+\lambda_2}{2}$ is a lattice point, and it belongs to D so by corollary 3 it is a vertex of D which it cannot be. $\square$

The next two lemmas exploit the central symmetries of a normal parallelotope tiling.

Lemma 5 *A dual cell of combinatorial dimension k, $0 < k \leq d$, is centrally symmetric if and only if the corresponding face of the tiling is the intersection of two parallelotopes. The dual cell and the corresponding face of the tiling have the same center of symmetry.*

Proof. Suppose that $F^{d-k} = P_1 \cap P_2$. Let c_1, c_2 be the centers of the parallelotopes P_1, P_2. The central symmetry $*$ at the point $c = \frac{c_1+c_2}{2}$ is a symmetry of the whole tiling and it maps F^{d-k} onto itself, so if a parallelotope P contains F^{d-k}, then so does $*(P)$. In terms of dual cells, if $v \in \mathrm{Vert}(D)$, then $*(v) \in \mathrm{Vert}(D)$. This proves the sufficiency.

To prove the necessity, consider a centrally symmetric dual cell D^k. Let $*$ be the central symmetry of D^k with center c. Note that

$$F^{d-k} = \bigcap_{v \in \mathrm{Vert}(D^k)} P(v)$$

so c is the center of symmetry of F^{d-k}. Let $v \in \mathrm{Vert}(D^k)$. Then the intersection of parallelotopes P_1 and P_2 centered at v and $*(v)$ respectively is exactly F^{d-k}. Indeed, c is the center of symmetry of both polytopes $P_1 \cap P_2$ and F^{d-k}, so their relative interiors intersect. Since the tiling is normal (face-to-face), $P_1 \cap P_2 = F^{d-k}$. This proves the necessity. $\square$

Two vertices v_1, v_2 of a centrally symmetric dual cell D are called *diametrically opposite* if $y_1 = *(y_2)$, where $*$ is the central symmetry transformation of D.

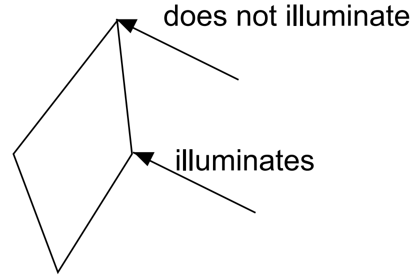
Lemma 6 *Let D be the dual cell corresponding to a face F of the tiling. Suppose that $X \subset \mathrm{Vert}(D)$ is a centrally symmetric nonempty subset. Then there is a centrally symmetric dual cell $D_1 \subset D$ with the same center of symmetry as X such that $X \subset \mathrm{Vert}(D_1)$.*

Proof. Consider the intersection F_1 of parallelotopes centered at points of X. The center of symmetry of X is also the center of symmetry of F_1, and of the tiling. Therefore the dual cell D_1 corresponding to F_1 is centrally symmetric. By construction, $X \subset \mathrm{Vert}(D_1)$. We also have $F \subset F_1$, therefore $D_1 \subset D$. $\square$

Lemma 7 *A centrally symmetric nonempty face of a dual cell D is a dual cell.*

Proof. Let D' be a face of D, invariant under a central symmetry $*$. Then $\mathrm{Vert}(D') = \mathrm{Vert}(D) \cap \mathrm{Vert}(*(D))$, therefore D' is a dual cell, for the reason that the dual cells form a combinatorial complex dual to the tiling. □

Definition 6 *A point v on the boundary of a (convex) polytope Q is said to be illuminated by a direction $u \in \mathrm{lin}(Q - Q)$, if for some $t > 0$ the point $v + tu$ belongs to the interior of Q.*



The minimal number $c(Q)$ of directions needed to illuminate the boundary of a polytope Q has been studied by several authors (see [Bol01]). It has been conjectured that 2^n directions are always sufficient for an n-dimensional polytope Q; and that $c(Q) = 2^n$ if and only if Q is affinely equivalent to the n-dimensional cube.

It is easy to see that if every vertex is illuminated by a direction from a given system, then the system illuminates all boundary points, so for every polytope Q, $\dim(Q) \geq 1$ we have $c(Q) \leq |\,\mathrm{Vert}(Q)|$.

Lemma 8 *Let $1 \leq k \leq d$. Consider a dual cell D^k and a projection h along the corresponding face F^{d-k} of the tiling onto a complementary k-space. Then $c(D^k) = c(h(D^k)) = |\,\mathrm{Vert}(D^k)|$. In other words, dual cells are* skinny, *see definition below.*

Proof. This directly follows from lemma 4. Let Q be any of the polytopes D^k, $h(D^k)$. Suppose that one direction u illuminates two distinct vertices v_1, v_2 of Q. Then u illuminates vertex v_1 of polytope $Q + (v_1 - v_2)$, so for some $\epsilon_1, \epsilon_2 > 0$ we have $v_1 + \epsilon_1 u \in \mathrm{relint}(Q)$, $v_1 + \epsilon_2 u \in \mathrm{relint}(Q + (v_1 - v_2))$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, then the point $v_1 + \epsilon u$ belongs to $\mathrm{relint}(Q)$ and to $\mathrm{relint}(Q + (v_1 - v_2))$. By lemma 4, these relative interiors do not intersect, unless $v_1 = v_2$. However, vertices v_1 and v_2 were chosen distinct. This contradiction proves the lemma. ∎

Lemma 9 *Suppose that* $\dim(D) = 3$*, where* D *is a dual cell. Then* $|\mathrm{Vert}(D)| \leq 8$*, with equality taking place if and only if* D *is affinely equivalent to a* 3*-cube.*

Proof. Assume without limiting the generality that 0 is a vertex of D. Consider the lattice $\Lambda' = \mathbb{Z}(\mathrm{Vert}(D))$. Rank of lattice Λ' is 3, because $\dim(D) = 3$. If $|\mathrm{Vert}(D)| \geq 9$, then $\mathrm{Vert}(D)$ contains two vertices from the same parity class in Λ', and hence from the same parity class in Λ. This contradicts lemma 5 on page 51. Therefore $|\mathrm{Vert}(D)| \leq 8$.

Now, D is a lattice polytope with respect to lattice Λ' and its images under the action of Λ' pack the space $\mathrm{lin}(\Lambda')$. Also, $|\mathrm{Vert}(D)| = 8$. It follows that D is a fundamental parallelepiped for the lattice Λ. ∎

Below we will call a polytope Q with $c(Q) = |\mathrm{Vert}(Q)|$ a *skinny* polytope.

Lemma 10 *Faces of dimension at least* 1 *of a skinny polytope are skinny.*

Proof. Enough to prove the lemma for facets and then propagate the result by induction. Suppose that Q is a skinny polytope and $F \subset Q$ is a facet. Suppose that directions $u_1, \ldots, u_k \in \mathrm{lin}(F - F)$ illuminate all vertices of F, where $k < |\mathrm{Vert}(F)|$.

Let n be an inward normal vector to the facet F. Then for small enough $\epsilon > 0$ the directions $u_i + \epsilon n$ illuminate all vertices $\mathrm{Vert}(F) \subset \mathrm{Vert}(Q)$ on the boundary of Q. We can use one direction for each of the remaining vertices $\mathrm{Vert}(Q) \setminus \mathrm{Vert}(F)$. It follows that Q is not skinny. $\square$

In fact, a more general result holds.

Lemma 11 *If Q is a skinny polytope and $X \subset \mathrm{Vert}(Q)$ is a subset with $|X| \geq 2$, then $conv(X)$ is skinny.*

Proof. Choose a point $y \in \mathrm{relint}(\mathrm{conv}(X))$ and take the face F of Q so that $y \in \mathrm{relint}(F)$. Face F is defined uniquely.

We prove that $\mathrm{conv}(X) \subset F$. If $F = Q$, then the result is immediate. Suppose that F is a proper face of Q. Let H be a hyperplane supporting face F on Q. Then $X \subset H$. Indeed, all points of X are vertices of Q so they belong to the same closed halfspace H^+ of H. On the other hand, if there is a point $y' \in \mathrm{conv}(X)$ which belongs to the interior of halfspace H^+, then (since $y \in \mathrm{relint}(\mathrm{conv}(X))$) there is a point on the line going through y and y' which belongs to the interior of the other halfspace H^-, which is a contradiction. Hence $\mathrm{conv}(X) \subset H \cap P = F$.

Since $\mathrm{conv}(X)$ and F are star sets with respect to y,[2] we have $\mathrm{relint}(\mathrm{conv}(X)) \subset \mathrm{relint}(F)$. If the number of directions sufficient to illuminate $\mathrm{conv}(X)$ is less than $|X|$, then we can use the same set of directions plus one direction for each vertex in $\mathrm{Vert}(F) \setminus X$ to illuminate F, and therefore F is not skinny. Using the previous lemma, we derive that Q is not skinny which is a contradiction. The lemma is proved. $\square$

[2] A set A is called star set with respect to y if any point in A can be connected with y by a line segment contained in A.

Lemma 12 *Let* $Q = [0,1]^n$, $n \geq 1$, $X \subset \text{aff}(Q)$, *so that* $\text{conv}(Q \cup X)$ *has vertex set* $\text{Vert}(Q) \cup X$. *Then* $\text{conv}(Q \cup X)$ *is* not *skinny.*

Proof. Let $x \in X$, t a vector so that $x + t \in \text{relint}(Q)$, chosen not to be parallel to any facet of Q. Then light direction t illuminates one of the vertices of Q, and hence one of the vertices of $\text{conv}(Q \cup \{x\})$, which means that the polytope $\text{conv}(Q \cup X)$ is not skinny. □

6.2 A lemma on the projections of polytopes

Consider a d-dimensional parallelotope P and the projection h along a linear space L^k onto a complementary linear space L^{d-k}. Call faces F^s of P with $L^k \subset \text{lin}(F^s - F^s)$ *commensurate* with the projection h. If F^s is commensurate, the polytope $h(F^s)$ has dimension $s - k$ and is a face of $h(P)$. Different commensurate faces of P have different images under h.

Lemma 13 *Let* M,N *be affine spaces,* $f : M \to N$ *an affine mapping,* $P \subset M$ *a convex set. Then* $\text{relint}(f(P)) = f(\text{relint}(P))$.

Proof. We may assume that $M = \text{aff}(P)$, $N = f(M)$. Then $N = \text{aff}(f(P))$. The set $f(\text{int}(P))$ is open, convex and dense in $f(P)$, which implies that $f(\text{int}(P)) = \text{int}(f(P))$. □

6.3 Dual cells affinely equivalent to cubes

In this section, we prove some properties of dual cells which are affinely equivalent to cubes of arbitrary dimension. Readers familiar with Delaunay tilings would note that some of these properties hold almost automatically, if you assume that the parallelotope tiling is a DV-tiling and the dual cells are the Delaunay polytopes.

First, a few notes on terminology. *Edge vectors* of a polytope are the vectors $v_1 - v_2$ for vertices v_1, v_2 such that $[v_1, v_2]$ is an edge. A k-dimensional nonempty proper face of a polytope is called primitive (in relation to the polytope) if it is contained in exactly $d - k$ facets of the polytope. By an s-cube we mean a cube of s dimensions.

Lemma 14 *Let D be a dual cell affinely equivalent to an s-cube. Then each face of D is also a subcells. All 1-subcells of D are edges of D.*

Proof. For every face $D' \subset D$ one can always pick a vector $\lambda \in \Lambda$ so that $D' = D \cap (D + \lambda)$. This, by statement 2 of lemma 4, implies that D' is a subcell of D. This proves the first statement of the lemma.

Next, if $[x, y]$ is a dual 1-cell, $x, y \in \mathrm{Vert}(D)$, let D' be the minimal face of D which contains both x and y. Suppose that $|\mathrm{Vert}(D')| > 2$. Then the face F' of the tiling dual to D' is of dimension less than $d - 1$, and $\frac{x+y}{2}$ is the center of symmetry of F'. However $\frac{x+y}{2}$ is also the center of symmetry of a facet corresponding to the dual 1-cell $[x, y]$ which is a contradiction proving the second statement. ∎

Definition 7 *Let F be a face of a parallelotope P of the tiling. The* associated collection *P_F of F on the parallelotope P is the set of all faces $F' \subset P$ such that $F + \lambda \subset F'$ for some $\lambda \in \Lambda$.*

For example, if F^{d-2} is a quadruple $(d-2)$-face of parallelotope P, then the associated collection $P_{F^{d-2}}$ contains 4 translated copies of F^{d-2}, 4 facets of P and P itself.

Lemma 15 *Let D be a dual cell affinely equivalent to an s-cube, $s \geq 1$. Let F be the corresponding face of the tiling, P a parallelotope of the tiling, $F \subset P$, and let $t_1, \ldots, t_s$ be linearly independent edge vectors of D. Let $f : \mathbb{R}^d \to M$ be a linear projection onto a linear space M such that* $\ker(f) \subset \operatorname{lin}(F - F)$. [3] *Then the following statements hold:*

1. $\dim(F) = d - s$
2. *All elements of P_F are primitive faces of P*
3. $\operatorname{lin}(F - F)$ *and* $\operatorname{lin}(D - D)$ *are complementary*
4. $\operatorname{aff}(D) \cap P \subset \bigcup_{G \in P_F} \operatorname{relint}(G)$
5. $f(\operatorname{aff}(D)) \cap f(P) \subset \bigcup_{G \in P_F} \operatorname{relint}(f(G))$
6. *The facet vectors of facets in P_F are $\{\pm t_1, \ldots, \pm t_s\}$*
7. *All subcells of D are faces of D.*
8. *If v is the center of P and $D = v + [0, t_1] \oplus \cdots \oplus [0, t_s]$, then $F - ([0, t_1] \oplus \cdots \oplus [0, t_s]) \subset P$, where symbol $\oplus$ stands for the direct Minkowski sum.*

Proof. Without limiting the generality, assume that 0 is a vertex of D, P is the parallelotope of the tiling with center 0, and $D = [0, t_1] \oplus \cdots \oplus [0, t_s]$. By lemma 14, the edges of D are its subcells. Let $F_1, \ldots, F_s$ be the facets of P corresponding to dual 1-cells $[0, t_1], \ldots, [0, t_s]$, and let $f_1, \ldots, f_s$ be the outward normals to those facets of P, so that facet F_j is defined by equation $f_j \cdot x = 1$. Let $L = \operatorname{lin}\{t_1, \ldots, t_s\}$.

[3] *By* ker *we denote the null space of a mapping.*

Inscribing a copy of D into the parallelotope. We can inscribe a copy of D into the parallelotope P by taking $E = \frac{1}{2}\sum_{i=1}^{s} t_i - D$. Proof of lemma 4 (page 46) describes this operation in detail: polytope E is denoted by S there.

Vertex $\frac{1}{2}\sum_{i=1}^{s} \epsilon_i t_i$ of E is the center of a $(d-s)$-face $F^{\epsilon_1 \dots \epsilon_s}$ in the associated collection P_F, where $\epsilon_i \in \{\pm 1\}$. Faces $F^{\epsilon_1 \dots \epsilon_s}$ are all the faces of P equivalent to F (in terms of the proof of lemma 4), in other words, all $(d-s)$-faces in the associated collection P_F.

Vertices $\frac{1}{2}\sum_{i \neq j} \epsilon_i t_i + \frac{1}{2} t_j$ and $\frac{1}{2}\sum_{i \neq j} \epsilon_i t_i - \frac{1}{2} t_j$ differ by t_j, the facet vector of F_j. Therefore the first point belongs to F_j, the second to $-F_j$. We have proved that

$$\frac{1}{2}\sum_{i=1}^{s} \epsilon_i t_i \in \epsilon_j F_j. \tag{6.7}$$

Suppose that $\epsilon_j = 1$. By varying ϵ_i for all $i \neq j$, we get only points of F_j. Therefore the vectors between these points are parallel to F_j, so we have $f_j \cdot t_i = 0$. Next, since t_j is a facet vector of F_j and equation $f_j \cdot x = 1$ defines the facet, we have $f_j \cdot t_j = 2$. Summarizing, we have

$$f_i \cdot t_j = 2\delta_{ij} \tag{6.8}$$

where δ_{ij} is the Kronecker delta. It follows that

$$E = L \cap \{x : |f_i(x)| \leq 1\} \tag{6.9}$$

Proof of statements 1 and 3. By lemma 14, there are s edges of D incident with vertex 0, so there are exactly s facets of P containing F, namely, $F_1, \dots, F_s$. The normal vectors to these facets are $f_1, \dots, f_s$. Therefore $\operatorname{lin}(F - F) = \cap_{i=1}^{s} \ker(f_i)$.

Since $\dim(F) = d - s$, the vectors $f_1, \ldots, f_s$ are linearly independent. Equation 6.8 implies that the linear space $\mathrm{lin}(F - F) = \cap_{i=1}^{s} \ker(f_i)$ is complementary to $\mathrm{lin}\{t_1, \ldots, t_s\} = \mathrm{lin}(D - D)$. This proves that $\dim(D) + \dim(F) = 1$.

Proof of statement 2. The cube $D^{\epsilon_1 \cdots \epsilon_s} = [0, \epsilon_1 t_1] \oplus \cdots \oplus [0, \epsilon_s t_s]$ is the dual cell corresponding to $F^{\epsilon_1 \cdots \epsilon_s}$. By lemma 14, there are exactly s 1-subcells of $D^{\epsilon_1 \cdots \epsilon_s}$ containing 0, which means that there are exactly s facets

$$\epsilon_1 F_1, \ldots, \epsilon_s F_s$$

of P which contain the face $F^{\epsilon_1 \cdots \epsilon_s}$. This means that $F^{\epsilon_1 \cdots \epsilon_s}$ is primitive: its dimension is equal to d minus the number of facets it belongs to. It implies that all faces which contain $F^{\epsilon_1 \cdots \epsilon_s}$ in their boundaries are primitive faces of P. Therefore all faces in P_F are primitive.

Proof of statement 4. We have $\mathrm{aff}(D) = L$, and

$$E = L \cap P \tag{6.10}$$

Indeed, on one hand, $E \subset L$ and $E \subset P$, on the other hand, P is bounded by inequalities $|f_i(x) \leq 1|$, $i = 1, \ldots, s$ and the same set of inequalities defines E in the space L, therefore $L \cap P \subset E$. We need to prove that $E \subset \bigcup_{G \in P_F} \mathrm{relint}(G)$. We will prove that the relative interior of each face of E is contained in the set on the right hand side.

Take a face E' of E. Its center is also the center of a uniquely defined dual cell D', $0 \in D' \subset D^{\epsilon_1 \cdots \epsilon_s}$. Let $F' \in P_F$ be the face of the tiling corresponding to D'. It has the same center of symmetry as D'.

We have $E' \subset P$. Since E' and F' share a center of symmetry, it follows that $E' \subset F'$ (draw a supporting hyperplane to face F' on P to prove it) and $\mathrm{relint}(E') \subset \mathrm{relint}(F')$. This proves statement 4.

Proof of statement 5. Without limiting the generality, we can assume that $L \subset M$, where M is the image space of the linear projection f. Then $f(\mathrm{aff}(D)) \cap f(P) = f(L) \cap f(P) = L \cap f(P)$. We have:

$$E = L \cap f(P). \tag{6.11}$$

Indeed, since $E \subset L$, $E \subset P$, and $f(E) = E$, we have $E \subset L \cap f(P)$. On the other hand, P is bounded by inequalities $|f_i(x)| \leq 1$, $i = 1, \ldots, s$, and E is equal to the set in equation 6.9, therefore $L \cap f(P) \subset E$.

We have $f(E) = E$. Let E' be a face of E. Since by the previous statement $\mathrm{relint}(E') \subset \mathrm{relint}(F')$ for some face $F' \in P_F$, we have

$$\mathrm{relint}(E') = f(\mathrm{relint}(E')) \subset f(\mathrm{relint}(F')), \tag{6.12}$$

which is equivalent to

$$\mathrm{relint}(E') = \mathrm{relint}(f(E')) \subset \mathrm{relint}(f(F')). \tag{6.13}$$

This establishes statement 5.

Proof of statement 6. We have enumerated all facets in the associated collection P_F: they are $\pm F_1, \ldots, \pm F_s$. Their facet vectors are $\{\pm t_1, \ldots, \pm t_s\}$. This proves statement 6.

Proof of statement 7. Sufficient to prove that all subcells of D which contain vertex 0 are faces of D. Since face F is a primitive face of parallelotope P, there are exactly 2^s faces of P in $\mathrm{St}(F)$. There are also exactly 2^s faces of D containing 0, and all of them are dual cells, by lemma 14. There are no other subcells of D containing 0, because there is a 1-1 correspondence between faces F' with $F \prec F' \prec P$ and dual cells D' with $0 \prec D' \prec D$.

Proof of statement 8. We need to prove that $F \oplus (-[0,t_1] \oplus \cdots \oplus [0,t_s]) \subset P$. We have

$$F \oplus (-[0,t_1] \oplus \cdots \oplus [0,t_s]) = \mathrm{conv}\left(\bigcup_{\epsilon_i \in \{0,1\}} (F - \sum_{i=1}^{s} \epsilon_i t_i)\right). \tag{6.14}$$

Indeed, the polytopes on the left and the right hand side have the same set of vertices. The polytope on the right hand side is contained in P since all polytopes $F - \sum_{i=1}^{s} \delta_i t_i$ are faces of P. This proves statement 8. $\square$

Lemma 16 *Let Q be a polytope, $\dim(Q) = n \geq 0$, $L \subset \mathrm{aff}(Q)$ an affine space, $L \cap \mathrm{relint}(Q) \neq \emptyset$. Then L intersects the relative interior of a face $F \subset Q$ where $\dim(F) \leq n - \dim(L)$.*

Proof. If $\dim(L) = 0$ or 1, then the result is trivial since L intersects P or the boundary of P respectively. This is the base of induction over $\dim(L)$. If $\dim(L) > 1$, consider a proper face F' of P such that $L \cap \mathrm{relint}(F') \neq \emptyset$. If $\dim(F') + \dim(L) \leq n$, the claim is proved. If $\dim(F') + \dim(L) > n$, the affine space $L' = L \cap \mathrm{aff}(F')$ intersects $\mathrm{relint}(F')$ and has dimension $\dim(L') \geq \dim(F') + \dim(L) - n$. By induction this means that L' intersects the relative interior of a face F'' of F' with $\dim(F'') \leq \dim(F') - \dim(L') \leq \dim(F') - \dim(F') - \dim(L) + n = n - \dim(L)$. Since F'' is a face of F', it is also a face of P. This completes the proof of the lemma. $\square$

6.4 Two subcells affinely equivalent to cubes

The following theorem and its corollary are key to the proof of our main result.

Theorem 11 *Let D be the dual cell corresponding to face F of the tiling. Let D_1, D_2 be subcells of D. Suppose that D_1, D_2 are affinely equivalent to cubes of dimensions* $\dim(D_1), \dim(D_2) \geq 1$. *Let h be the projection along F onto a complementary linear space M.*

1. *If* $\mathrm{aff}(D_1) \cap \mathrm{aff}(D_2) \neq \emptyset$, *then $D_1 \cap D_2$ is a nonempty face of both D_1 and D_2 and* $\mathrm{aff}(D_1 \cap D_2) = \mathrm{aff}(D_1) \cap \mathrm{aff}(D_2)$.

2. *If* $\mathrm{aff}(D_1) \cap \mathrm{aff}(D_2) = \emptyset$ *and* $\dim(D_1) + \dim(D_2) \geq \mathrm{combdim}(D)$, *then D_1, D_2 have at least* $\dim(D_1) + \dim(D_2) - \mathrm{combdim}(D) + 1$ *linearly independent common edge directions.*

3. *Projection h restricted to* $\mathrm{aff}(D_1 \cup D_2)$ *is 1-1.*

4. *The number of linearly independent common edge directions of D_1 and D_2 is equal to* $\dim(\mathrm{lin}(D_1 - D_1) \cap \mathrm{lin}(D_2 - D_2))$.

Proof. Let $\Lambda_n = \mathbb{Z}_{\mathrm{aff}}(\mathrm{Vert}(D_n))$, $K_n = \mathrm{aff}(D_n)$, and let F_n be the face of the tiling corresponding to the dual cell D_n, $n = 1, 2$. Let P be the polytope of the tiling whose center is 0, $d_n = \dim(D_n)$. Let $T_n = \{\pm t_1^n, \ldots, \pm t_{d_n}^n\}$ where $t_1^n, \ldots, t_{d_n}^n$ are the edge vectors of D_n (we have $|T_n| = 2d_n$), v_n a vertex of D_n so that $D_n = v_n + [0, t_1^n] \oplus \cdots \oplus [0, t_{d_n}^n]$. Let C_n be the convex hull of $0, 1$-combinations of $t_1^n, \ldots, t_{d_n}^n$, ie. $C_n = [0, t_1^n] \oplus \cdots \oplus [0, t_{d_n}^n]$. C_n is a dual cell, a translate of D_n. Let $\Delta_n = \mathbb{Z}(T_n)$, $L_n = \mathrm{lin}(T_n)$. The symbol $\mathbb{Z}(A)$ denotes the lattice generated by A, that is, the set of all finite linear combinations of elements of A; $\mathbb{Z}_{\mathrm{aff}}(A)$ denotes the affine lattice generated by set A.

For $\lambda \in \Lambda$, $P(\lambda)$ will stand for the parallelotope of the tiling whose center is λ. Let F_n be the face of the tiling corresponding to the dual cell D_n, $n = 1, 2$. Consider sets

$$\begin{aligned} Q_n &= F_n + \mathrm{lin}(D_n - D_n), \\ U_n &= \bigcup_{\lambda \in \Lambda_n} P(\lambda), \\ n &= 1, 2, \end{aligned} \tag{6.15}$$

We begin by proving supplementary results (A) - (E).

(A) *The following statements are equivalent:*

1. $\Lambda_1 \cap \Lambda_2 \neq \emptyset$

2. $K_1 \cap K_2 \neq \emptyset$

3. $\mathrm{int}(Q_1) \cap \mathrm{int}(Q_2) \neq \emptyset$

4. $\mathrm{int}(U_1) \cap \mathrm{int}(U_2) \neq \emptyset$

Proof. Indeed, we have: $\Lambda_n \subset K_n \subset \mathrm{int}(Q_n) \subset \mathrm{int}(U_n)$, $n = 1, 2$. The first inclusion is trivial. We need to prove the statement $K_n \subset \mathrm{int}(Q_n)$. By statement 3 of lemma 15, the linear spaces $\mathrm{lin}(F_n - F_n)$ and $\mathrm{lin}(D_n - D_n)$ are complementary. Let c_n be the center of symmetry of F_n. We have $c_n \in \mathrm{relint}(F_n)$, therefore for every $x \in \mathrm{lin}(D_n - D_n)$, $c_n + x \in \mathrm{relint}(F_n) + \mathrm{lin}(D_n - D_n) = \mathrm{int}(\mathrm{lin}(D_n - D_n) + F_n) = \mathrm{int}(Q_n)$. We conclude that $K_n = \mathrm{aff}(D_n) = \mathrm{lin}(D_n - D_n) + c_n \subset \mathrm{int}(Q_n)$. To prove the last inclusion,

$\mathrm{int}(Q_n) \subset \mathrm{int}(U_n)$, note that by statement 8 of lemma 15, the following inclusion holds: $F_n - C_n \subset P(v_n)$, so

$$\begin{aligned} Q_n = F_n + \mathrm{lin}(D_n - D_n) = F_n - C_n + \mathbb{Z}(t_1^n, \ldots, t_{d_n}^n) \subset \\ P(v_n) + \mathbb{Z}(t_1^n, \ldots, t_{d_n}^n) = P + v_n + \mathbb{Z}(t_1^n, \ldots, t_{d_n}^n) = \\ P + \Lambda_n = U_n. \end{aligned} \tag{6.16}$$

This implies that $\mathrm{int}(Q_n) \subset \mathrm{int}(U_n)$.

The three inclusions prove that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$. Finally if $x \in \mathrm{int}(U_1) \cap \mathrm{int}(U_2)$, then there is a number $\epsilon > 0$ so that $B_\epsilon(x) \subset U_2$. Since $x \in U_1$, there is a point $\lambda \in \Lambda_1$ so that $x \in P(\lambda)$. Take $y \in \mathrm{int}(P(\lambda))$, $|y - x| < \epsilon$. By choice of ϵ, $y \in U_2$. There is a point $\lambda' \in \Lambda_2$ so that $y \in P(\lambda')$. We therefore have $\mathrm{int}(P(\lambda)) \cap P(\lambda') \neq \emptyset$. From the face-to-face property of the tiling it follows that $P(\lambda) = P(\lambda')$, hence $\lambda = \lambda'$ and $\Lambda_1 \cap \Lambda_2 = \emptyset$. This proves that $4 \Rightarrow 1$. $\square$

(B) *The following numbers are equal:* $\mathrm{rank}(T_1 \cap T_2)$*,* $\dim(L_1 \cap L_2)$*,* $d - \min\{\dim(G) : G \in P_{F_1} \cap P_{F_2}\}$.

Proof. We prove this in a chain of inequalities. Obviously $\mathrm{rank}(T_1 \cap T_2) \leq \dim(L_1 \cap L_2)$. Next, let $L = L_1 \cap L_2$. By lemma 16 L intersects the relative interior of a face G of P with $\dim(G) \leq d - \dim(L)$. By statement 4 of lemma 15, since $L \subset L_1$, $G \in P_{F_1}$. Similarly, since $L \subset L_2$, $G \in P_{F_2}$. Thus $\dim(L_1 \cap L_2) = \dim(L) \leq d - \min\{\dim(G) : G \in P_{F_1} \cap P_{F_2}\}$.

Finally let $G \in P_{F_1} \cap P_{F_2}$ be a face of minimal dimension k. By the definition of associate collection, all facets which contain G belong to both P_{F_1} and P_{F_2}, and by statement 6 of lemma 15 the facet vectors corresponding to those facets belong to T_1

and to T_2. Since by statement 2 of lemma 15 G is a primitive face of P, G belongs to exactly $d-k$ facets of P. Hence $P_{F_1} \cap P_{F_2}$ contains at least $d-k$ pairs of parallel facets of P and $\mathrm{rank}(T_1 \cap T_2) = \frac{1}{2}|T_1 \cap T_2| \geq d-k = d - \dim(G) = d - \min\{\dim(G) : G \in P_{F_1} \cap P_{F_2}\}$ which closes the chain of inequalities. $\square$

(C) $\mathbb{Z}(T_1) \cap \mathbb{Z}(T_2) = \mathbb{Z}(T_1 \cap T_2)$.

Proof. We have $\dim(\mathrm{lin}(L_1 \cup L_2)) = \dim(L_1) + \dim(L_2) - \dim(L_1 \cap L_2)$, $\dim(L_1) = \frac{1}{2}|T_1|$, $\dim(L_2) = \frac{1}{2}|T_2|$, and $\mathrm{rank}(T_1 \cap T_2) = \frac{1}{2}|T_1 \cap T_2| = \dim(L_1 \cap L_2)$. Together with $|T_1 \cup T_2| = |T_1| + |T_2| - |T_1 \cap T_2|$ this implies that $\dim(\mathrm{lin}(L_1 \cup L_2)) = \frac{1}{2}|T_1 \cup T_2|$. This means that an orientation of $T_1 \cup T_2$ (a set of vectors A with $\frac{1}{2}|T_1 \cup T_2| = |A|$ and $T_1 \cup T_2 = A \cup -A$) is a basis for $\mathrm{lin}(L_1 \cup L_2)$. Let $A_1 = A \cap T_1$, $A_2 = A \cap T_2$. A_1, A_2 are bases for $\mathbb{Z}(T_1)$, $\mathbb{Z}(T_2)$ respectively.

We clearly have $\mathbb{Z}(T_1 \cap T_2) \subset \mathbb{Z}(T_1) \cap \mathbb{Z}(T_2)$. On the other hand, take $z \in \mathbb{Z}(T_1) \cap \mathbb{Z}(T_2)$. Since $z \in \mathbb{Z}(T_1)$, we have $z = \sum_{a \in A_1} z_a a$. Similarly $z = \sum_{a \in A_2} z'_a a$. However, A is a basis for $L_1 \cup L_2$, therefore these two representations are the same, so $z_a = 0$ for $a \in A_1 \setminus A_2$, $z'_a = 0$ for $a \in A_2 \setminus A_1$, and $z_a = z'_a$ for $a \in A_1 \cap A_2$. Hence $z \in \mathbb{Z}(A_1 \cap A_2) = \mathbb{Z}(T_1 \cap T_2)$. $\square$

(D) $\dim(h(L_1) \cap h(L_2)) = \dim(L_1 \cap L_2)$.

Proof. Let $N = h(L_1) \cap h(L_2)$. By lemma 16 N intersects the relative interior of a face A of $h(P)$ with $\dim(A) \leq d - \dim(F) - \dim(N)$. By statement 5 of lemma 15, since $N \subset h(L_1)$, $A \in h(P_{F_1})$. But $N \subset h(L_2)$, therefore $A \in h(P_{F_2})$. Thus $\dim(h(L_1) \cap h(L_2)) = \dim(N) \leq d - \dim(F) - \min\{\dim(X) : X \in h(P_{F_1}) \cap h(P_{F_2})\}$.

From chapter 6.2 and paragraph (B) of this proof, we get

$$\begin{aligned} d - \dim(F) - \min\{\dim(X) : X \in h(P_{F_1}) \cap h(P_{F_2})\} = \\ d - \min\{\dim(G) : G \in P_{F_1} \cap P_{F_2}\} = \dim(L_1 \cap L_2). \end{aligned} \tag{6.17}$$

(since all faces in P_{F_1} and P_{F_2} are commensurate with the projection f).

This produces $\dim(h(L_1) \cap h(L_2)) \leq \dim(L_1 \cap L_2)$. On the other hand, $L_1 \cap L_2 \subset L_1$ and the null space of projection f is contained in $\mathrm{lin}(F_1 - F_1)$, which is complementary to L_1 by lemma 15, statement 3. Therefore $\dim(h(L_1 \cap L_2)) = \dim(L_1 \cap L_2)$. We also have $h(L_1 \cap L_2) \subset h(L_1) \cap h(L_2)$, so, combining these two results, we have $\dim(L_1 \cap L_2) \leq \dim(h(L_1) \cap h(L_2))$.

We have proved that $\dim(h(L_1) \cap h(L_2)) = \dim(L_1 \cap L_2)$. □

(E) *Mapping* f *acts on the linear space* $\mathrm{lin}(L_1 \cup L_2)$ *bijectively.*

Proof. From D, we have

$$\begin{aligned} \dim(h(\mathrm{lin}(L_1 \cup L_2))) = \\ \dim(\mathrm{lin}(h(L_1 \cup L_2))) = \\ \dim(\mathrm{lin}(h(L_1) \cup h(L_2))) = \\ \dim(h(L_1)) + \dim(h(L_2)) - \dim(h(L_1) \cap h(L_2)) = \\ \dim(L_1) + \dim(L_2) - \dim(L_1 \cap L_2) = \\ \dim(\mathrm{lin}(L_1 \cup L_2)). \end{aligned}$$

This proves our statement. □

We are now ready to prove the theorem. First we prove statement 1. We are given the condition that $K_1 \cap K_2 \neq \emptyset$. Then, by the chain of equivalences proved above, $\Lambda_1 \cap \Lambda_2 \neq \emptyset$. Without limitation of generality we may assume that $0 \in \Lambda_1 \cap \Lambda_2$, so $\Lambda_n = \mathbb{Z}(T_n)$, $K_n = L_n$, $\dim(K_1 \cap K_2) = \operatorname{rank}(\Lambda_1 \cap \Lambda_2)$. The lattice $\Lambda_1 \cap \Lambda_2$ is generated by $T_1 \cap T_2$.

We now prove that the affine cubes D_1 and D_2 have a common face of dimension $\dim(K_1 \cap K_2)$.

Before giving a general proof, we consider a special case when $\dim(K_1) = \dim(K_2) = 2$, affine spaces K_1 and K_2 intersect over a point z, and parallelograms are located as shown in figure 6.1. We are proving that such a configuration is impossible. Choose vertices $v_1 \in D_1$ and $v_2 \in D_2$ so that vectors $v_1 - z$ and $v_2 - z$ illuminate vertices v_1, v_2 of parallelograms D_1, D_2 respectively. For some $\epsilon > 0$ we have $v_1 + \epsilon(v_1 - z) \in \operatorname{relint}(D_1)$ and $v_2 + \epsilon(v_2 - z) \in \operatorname{relint}(D_2)$. The quadrilateral Q with vertices v_1, v_2, $v_1 + \epsilon(v_1 - z)$, $v_2 + \epsilon(v_2 - z)$ is shown on the right hand side of the picture. We then have $\operatorname{relint}(Q) \subset \operatorname{relint}(\operatorname{conv}(D_1 \cup D_2)))$. This is because affine spaces K_1, K_2 are complementary. From the right hand side of the picture, we can also see that $\frac{1}{2}(v_1 - z + v_2 - z)$ illuminates both vertices v_1, v_2 on Q and, consequently, on $\operatorname{conv}(D_1 \cup D_2)$ which contradicts the fact that polytope D^4 is skinny (ie., no light direction illuminates two vertices) and that the convex hull of a subset in $\operatorname{Vert}(D^4)$ of dimension at least 1 is also skinny.

Now we can prove statement 1 in the general case. Let $K = K_1 \cap K_2 = L_1 \cap L_2$. The (unoriented) basis in K can be chosen as $T_1 \cap T_2$. If at least one of D_1 and D_2 intersects K, eg. $D_1 \cap K \neq \emptyset$, then $D_1 \cap K$ is a face of D_1. By lemma 11 on page 55, polytope $\operatorname{conv}((D_1 \cap K) \cup D_2)$ is skinny, and by lemma 12 we should $D_1 \cap K \subset D_2$

(otherwise the polytope is not skinny).

It follows that $D_2 \cap K \neq \emptyset$, therefore by a similar argument $D_2 \cap K \subset D_1$. We have $D_1 \cap K = D_2 \cap K = D_1 \cap D_2$. Since $D_1 \cap D_2$ is affinely equivalent to a cube with the same edge set $T_1 \cap T_2$, we have proved statement 1.

Suppose now that $D_1 \cap K = D_2 \cap K = \emptyset$. We prove that it is impossible by switching to certain faces D_1'', D_2'' of cubes D_1, D_2 to come to a contradiction. We will observe that $\mathrm{conv}(D_1'' \cup D_2'')$ is not skinny, like in the example we've just considered.

We have $K_1 \not\subset K_2$ and $K_2 \not\subset K_1$, therefore $T_2 \setminus T_1 \neq \emptyset$ and $T_1 \setminus T_2 \neq \emptyset$. Let D_1' be any face of D_1 whose set of edge vectors is $T_1 \setminus T_2$, and let $D_2' = D_2$. Since $T_1 \setminus T_2 \neq \emptyset$, we have $\dim(D_1') > 0$. Since the set of edge vectors of D_1' is $T_1 \setminus T_2$, the set of edge vectors of D_2' is T_2, we know that the affine lattices $\mathbb{Z}_{\mathrm{aff}}(\mathrm{Vert}(D_1'))$, $\mathbb{Z}_{\mathrm{aff}}(\mathrm{Vert}(D_2'))$ are translates of lattices $\mathbb{Z}(T_1 \setminus T_2)$ and $\mathbb{Z}(T_2)$ by vectors from $\mathbb{Z}(T_1 \cup T_2)$.

In paragraph (C) we proved that $\mathrm{rank}(T_1 \cup T_2) = \frac{1}{2}|T_1 \cup T_2|$, hence the lattices $\mathbb{Z}(T_1 \setminus T_2)$ and $\mathbb{Z}(T_2)$ are complementary in $\mathbb{Z}(T_1 \cup T_2)$.

It follows that the intersection of $\mathbb{Z}_{\mathrm{aff}}(\mathrm{Vert}(D_1'))$ and $\mathbb{Z}_{\mathrm{aff}}(\mathrm{Vert}(D_2'))$ consists of one point, which we assume to be 0 (without limiting the generality). Note that $0 \in K$, and that linear spaces $\mathrm{lin}(D_1' - D_1')$ and $\mathrm{lin}(D_2' - D_2')$ have a trivial intersection.

Next, for $n = 1, 2$ let D_n'' be the face of D_n' of minimal dimension whose affine hull contains 0. Since we have assumed that $D_n \cap K = \emptyset$, 0 cannot be a vertex of D_n, therefore $\dim(D_n'') > 0$. We can choose a vertex v_n of D_n'' so that light direction v_n illuminates the boundary of polytope D_n'' at v_n. It follows that light direction $v_1 + v_2$ illuminates both vertices v_1, v_2 on $\mathrm{conv}(D_1'' \cup D_2'')$. But the polytope $\mathrm{conv}(D_1'' \cup D_2'')$ is skinny by lemma 11, which is the desired contradiction.

Next we prove statement 2 of the theorem. We are given the condition $K_1 \cap K_2 = \emptyset$. Then $\mathrm{int}(Q_1) \cap \mathrm{int}(Q_2) = \emptyset$. Since the sets Q_1, Q_2 are convex, there is a hyperplane H separating them. We have $F \subset F_1 \cap F_2$, hence $F \subset H$. The affine spaces K_1, K_2 are contained in the interiors of Q_1 and Q_2 respectively, so they belong to different open half-spaces defined by H.

Hyperplane $h(H)$ in M strictly separates $h(K_1)$ and $h(K_2)$ (this means they are contained in different open half-spaces of $h(H)$), because $F \subset H$ and h is the projection along F onto the complementary linear space M.

Since $F \subset F_n$ and F_n is complementary to K_n, $\dim(K_n) = \dim(h(K_n))$. Let $s = \dim(h(K_1)) + \dim(h(K_2)) - \dim(M)$. Since $\dim(M) = \mathrm{combdim}(D)$, by the hypothesis of the theorem we have $s \geq 0$. Since $h(K_1) \cap h(K_1) = \emptyset$, linear spaces $h(L_1) = h(K_1) - h(K_1)$ and $h(L_2) = h(K_2) - h(K_2)$ have at least $s+1$-dimensional intersection, ie. $\dim(h(L_1) \cap h(L_2)) \geq s+1$. We have proved in paragraph (D) that $\dim(h(L_1) \cap h(L_2)) = \dim(L_1 \cap L_2)$, and in paragraph (B) that $\dim(L_1 \cap L_2) = \mathrm{rank}(T_1 \cap T_2) = \frac{1}{2}|T_1 \cap T_2|$. Therefore D_1 and D_2 have at least $s+1$ linearly independent edge directions.

To prove statement 3, note that it is equivalent to the following:

$$\dim(\mathrm{aff}(K_1 \cup K_2)) = \dim(\mathrm{aff}(h(K_1) \cup h(K_2))). \tag{6.18}$$

If $K_1 \cap K_2 \neq \emptyset$, then this statement was already proved in paragraph (E). If $K_1 \cap K_2 = \emptyset$, then $\dim(\mathrm{aff}(K_1 \cup K_2)) = 1 + \dim(\mathrm{lin}(L_1 \cup L_2)) = 1 + \dim(\mathrm{lin}(h(L_1) \cup h(L_2))) = \dim(\mathrm{aff}(h(K_1) \cup h(K_2)))$. The middle equality is found in paragraph (E).

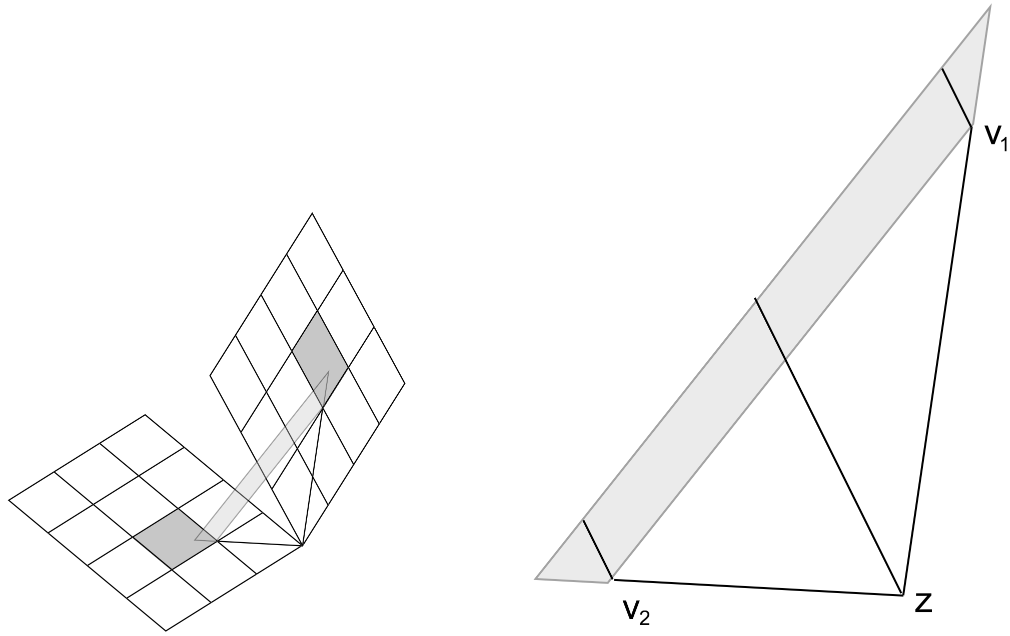


Figure 6.1: Two parallelograms in complementary 2-spaces

Statement 4 directly follows from the result of paragraph (B), that $\dim(L_1 \cap L_2) = \operatorname{rank}(T_1 \cap T_2)$. This completes the proof of the theorem. $\square$

Corollary 6 *Let D^4 be a dual 4-cell corresponding to a face F of the tiling. Then two distinct parallelogram subcells $\Pi_1, \Pi_2 \subset D^4$ are either*

1. *complementary:* $\Pi_1 \cap \Pi_2 = \{v\}$ *and* $\dim \operatorname{aff}(\Pi_1 \cup \Pi_2) = 4$,
2. *adjacent:* $\Pi_1 \cap \Pi_2 = I$, *where I is an edge (a dual combinatorial 1-cell) and* $\dim \operatorname{aff}(\Pi_1 \cup \Pi_2) = 3$,
3. *translate:* $\Pi_1 = \Pi_2 + t$ *where* $t \notin \operatorname{lin}(\Pi_1 - \Pi_1)$ *and* $\dim \operatorname{aff}(\Pi_1 \cup \Pi_2) = 3$, *or*
4. *skew:* $\Pi_1 \cap \Pi_2 = \emptyset$, *there is exactly one pair of edges of* Π_1 *parallel to a pair of edges of* Π_2, *and* $\dim \operatorname{aff}(\Pi_1 \cup \Pi_2) = 4$.

Moreover, if f is the projection along F onto a complementary 4-space, then f is 1-1 on the affine space $\operatorname{aff}(\Pi_1 \cup \Pi_2)$.

Chapter 7

Proof of theorem 1

In this chapter, we complete the proof of our main result:

Theorem 1 *The Voronoi conjecture is true for* 3*-irreducible parallelotope tilings.*

By theorems 5 and 8 (pages 20 and 35), proving the theorem is equivalent to checking that each parallelogram dual cell is coherent with respect to each dual 4-cell that contains it.

Let D^4 be a dual cell in a 3-irreducible tiling. We will prove that the collection $\mathcal{R}$ of incoherent parallelogram subcells in D^4 is empty. If $\mathcal{R}$ is not empty, what can we say about it?

- Each vertex of a parallelogram in $\mathcal{R}$ belongs to at least one other parallelogram in $\mathcal{R}$ (theorem 9 on page 38).
- Each pair of parallelograms in $\mathcal{R}$ is either complementary, adjacent, translate, or skew (corollary 6 on page 71).

We suppose that $|\mathcal{R}| \geq 1$. In each of the following cases, which cover all possibilities, we will get a contradiction.

1. $|\mathcal{R}| = 1$

2. D^4 is asymmetric, $|\mathcal{R}| \geq 2$ and at least one pair of parallelograms in $\mathcal{R}$ is adjacent, translate, or skew.

3. D^4 is centrally symmetric and $|\mathcal{R}| \geq 2$.

4. D^4 is asymmetric, $|\mathcal{R}| \geq 2$ and all pairs of parallelograms in $\mathcal{R}$ are complementary.

Case 1 is the easiest; in fact, its impossibility follows directly from theorem 9 on page 38. Cases 2 and 3 are similar in that there is some extra geometric information available about D^4. Either we know that D^4 is centrally symmetric, or that the system of parallelograms in D^4 has a "singularity" (a pair of adjacent, translate, or skew parallelograms).

In case 4, we do not have this much geometric information, however we have very homogeneous combinatorial conditions on the system of parallelograms $\mathcal{R}$. Namely, (1) each two parallelograms intersect over exactly one vertex, and (2) each vertex of a parallelogram belongs to at least one other parallelogram. It allows for some graph-theoretic methods to be applied.

The next section proves some technical lemmas. The following four sections deal with cases 1-4.

7.1 Three technical lemmas

Lemma 17 *Let C be a triangular prism. Let x_1, x_2, x_3 be distinct vertices of C. Then either* $\operatorname{conv}\{x_1, x_2, x_3\}$ *is a triangular facet of C, or among x_1, x_2, x_3 there are two diagonally opposite vertices of a parallelogram face of C.*

Proof is obtained by inspection. □

Lemma 18 *If D^3, D are dual cells, $D^3 \subset D$, C a triangular prism with* $\mathrm{Vert}(D^3) \subset \mathrm{Vert}(C) \subset \mathrm{Vert}(D)$, *then $D^3 = C$.*

Proof. Note that a diagonal I of a parallelogram face Π of C cannot be a dual edge, because by lemma 6 there is a dual subcell $D' \subset D$ with $\mathrm{Vert}(\Pi) \subset \mathrm{Vert}(D')$, with the same center of symmetry as Π. Since $\mathrm{combdim}(D') \geq 2$, the corresponding face of the tiling is of dimension at most $(d-2)$. However, it has the same center of symmetry as the $(d-1)$-face corresponding to I, which is a contradiction.

Now we can prove the lemma. Since $\mathrm{Vert}(D^3) \subset \mathrm{Vert}(C)$, we have $|\,\mathrm{Vert}(D^3)| \leq 6$, so D^3 can only be a triangular prism, a simplex, an octahedron, or a pyramid over a parallelogram. The octahedron is excluded because it has the same number of vertices as C, but is not a triangular prism.

If D^3 is either a simplex or a pyramid, then by the previous lemma combined with the argument at the beginning of this proof, each triangular face of D^3 is a face of C, but C has only 2 triangular faces whereas D^3 has 4. Therefore D^3 is a prism, so $D^3 = C$. □

Lemma 19 *Consider a 3-irreducible tiling. Let D^4 be a dual 4-cell. Suppose that there is a triangular prism C with* $\mathrm{Vert}(C) \subset \mathrm{Vert}(D^4)$. *Suppose also that at least one parallelogram face of C is a subcell of D^4, and each parallelogram face of C is either a dual cell, or is inscribed into an octahedral dual cell.*

Then the following statements hold:

(I) Not all parallelogram faces of C are dual cells.

(II) Moreover, exactly 2 *parallelogram faces of* C *are subcells,* $|\,\mathrm{Vert}(D^4)|\;=\;8$ *and* D^4 *is affinely equivalent to the following polytope (columns are coordinates of vertices):*

$$\mathrm{conv}\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \tag{7.1}$$

Proof. Let T_1, T_2 be triangular faces of C, Π_1, Π_2, Π_3 parallelogram faces.

Note that all edges of C are dual cells. Indeed, each parallelogram face Π_k of C can be inscribed into a centrally symmetric subcell D of D^4 (by lemma 6). By the condition of the lemma, either $D = \Pi_k$, or D is an octahedron. In each case, edges of Π_k are dual cells.

We prove that T_1, T_2 are dual cells. Indeed, all edges of C are dual cells, in particular the edges of triangles T_1, T_2 are dual 1-cells. Now let t be a vector in the lattice of the tiling with $T_1 = T_2 + t$. We have $T_1 \subset D^4 \cap (D^4 + t)$. By lemma 4, $D = D^4 \cap (D^4 + t)$ is a dual 2-or 3-cell. If D is a dual 2-cell, then it can only be the triangle T. If D is a dual 3-cell, then D is either a simplex, an octahedron, a triangular prism, or a pyramid over a parallelogram. In each case, a cycle of 3 edges on D is the boundary of a dual 2-cell, so T is a subcell of D and therefore a dual cell.

(A). *Definition of the complex* K*.* Consider the following abstract complex:

$$\partial D^4 = \{D \subset D^4 : \text{ D is a subcell }, D \neq D^4\} \tag{7.2}$$

By definition, this complex is dual to $\mathrm{St}(F^{d-4}) \setminus \{F^{d-4}\}$, where F^{d-4} is the face of the tiling corresponding to D^4. Therefore it is combinatorially isomorphic to a polyhedral

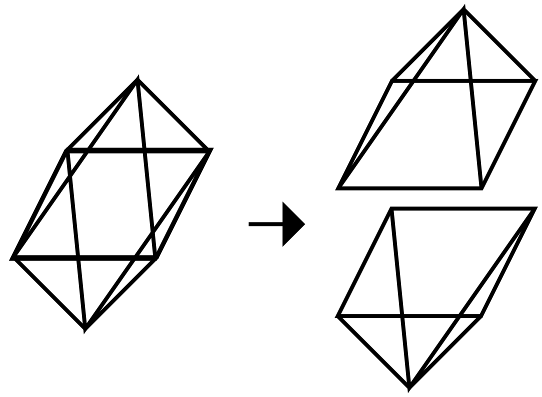


Figure 7.1: Splitting the octahedral cell

subdivision of the 3-sphere.

We will now define complex K as a subdivision of ∂D^4, in the following way.

Each of the vertex sets $\mathrm{Vert}(\Pi_k)$ is a centrally symmetric subset of $\mathrm{Vert}(D^4)$, hence by lemma 6 there is a centrally symmetric dual subcell $E_k \subset D^4$ with $\mathrm{Vert}(\Pi_k) \subset \mathrm{Vert}(E_k)$. By the assumption in the statement of the lemma, either

1. $E_k = \Pi_k$, or
2. E_k is an octahedron (we say that Π_k is inscribed into the octahedron E_k).

If E_k is an octahedron, then we subdivide it into two pyramids with base Π_k, as shown in figure 7.1).

Like ∂D^4, the new complex K is combinatorially isomorphic to the subdivision of the 3-dimensional sphere. Complex K is important to us because faces of C are present among its cells.

(B). *Application of Alexander's separation theorem.* We now use the following theorem:

Theorem 12 *(Corollary on page 338 of paper [Ale22]) Any closed* 2*-chain in* K *bounds exactly two open* 3*-chains. Moreover, these two* 3*-chains have only the elements of the* 2*-chain in common.*

We define 2-chain B in the complex K as the sum of the cellular 2-chains defined by faces of C, namely, T_1, T_2, Π_1, Π_2, Π_3. Chain B is closed. Therefore, by theorem 12 above, B bounds exactly two open 3-chains in K, K_1 and K_2 where $K_1 + K_2 \equiv K \pmod 2$. Moreover, any common element of K_1 and K_2 is contained in B.

(C). *Counting vertices.* We define sets U, V of vertex pairs. If D^4 is centrally symmetric, then

$$\begin{aligned} U &= \{(y_1, y_2) \in \mathrm{Vert}(K_1 \setminus B) \times \mathrm{Vert}(K_2 \setminus B) : y_1 \neq *(y_2)\}, \\ V &= \{(y_1, y_2) \in \mathrm{Vert}(K_1 \setminus B) \times \mathrm{Vert}(K_2 \setminus B) : y_1 = *(y_2)\} \end{aligned} \tag{7.3}$$

where $*$ is the central symmetry of D^4. If D^4 is asymmetric, then we define

$$\begin{aligned} U &= \mathrm{Vert}(K_1 \setminus B) \times \mathrm{Vert}(K_2 \setminus B), \\ V &= \emptyset \end{aligned} \tag{7.4}$$

The following relationship establishes a 1-1 correspondence between set U and the collection of those parallelograms Π_k which are inscribed into octahedra E_k:

$$\begin{gathered} (y_1, y_2) \to \Pi_k \\ \text{if and only if} \\ E_k = \mathrm{conv}(\Pi_k \cup \{y_1, y_2\}). \end{gathered} \tag{7.5}$$

First, we prove that for each pair $(y_1, y_2) \in U$, there is a parallelogram Π_k which satisfies condition 7.5. Each pair $(y_1, y_2) \in U$ is a diagonal pair of vertices of a centrally symmetric dual cell $D \subset D^4$, $D \neq D^4$. However, there is no element of complex K which contains both y_1 and y_2, for, if $y_1, y_2 \in D^3$, then, since $K \equiv K_1 + K_2 \pmod 2$, either $D^3 \in K_1$ or $D^3 \in K_2$. In the first case, point y_2 does not belong to $(K_2 \setminus B)$, because $(K_2 \setminus B) \cap K_1 = \emptyset$. The second case is treated similarly. Therefore y_1 and y_2 can only be diametrically opposite vertices of an octahedron E_k. Since y_1, y_2 were chosen not to belong to B, we have and $y_1, y_2 \notin \mathrm{Vert}(\Pi_k)$, so $E_k = \mathrm{conv}(\Pi_k \cup \{y_1, y_2\})$.

Next, we prove that each parallelogram Π_k, inscribed into octahedral dual cell E_k, corresponds to some pair of vertices $(y_1, y_2) \in U$. If a parallelogram Π_k extends to the octahedron E_k, then the two points $\{y_1, y_2\} = \mathrm{Vert}(E_k) \setminus \mathrm{Vert}(\Pi_k)$ cannot be vertices of C (this is checked directly, using the fact that a diagonal of a parallelogram face of C cannot be a dual 1-cell).

Pyramids $\mathrm{conv}(\Pi_k \cup \{y_1\})$ and $\mathrm{conv}(\Pi_k \cup \{y_2\})$ are elements of complex K. They belong to different chains K_1, K_2, because the cell Π_k belongs to the common boundary B of K_1 and K_2. Hence, renumbering, if necessary, we have $y_1 \in K_1 \setminus B$, $y_2 \in K_2 \setminus B$. If $*$ is the central symmetry of D^4, then we have $y_1 \neq *(y_2)$ because y_1, y_2 are diametrically opposite vertices of a centrally symmetric 3-subcell E_k of D^4.

Therefore $|U|$ is equal to the number of parallelogram faces of C which are inscribed into octahedral dual cells and $|U| \leq 2$. Let $N_i = |\,\mathrm{Vert}(K_i \setminus B)|$, $k = 1, 2$. We have

$$
\begin{aligned}
|U| + |V| &= N_1N_2, \\
|V| &\leq \min(N_1, N_2), \\
N_1N_1 \geq |U| &\geq N_1N_2 - \min(N_1, N_2), \\
|U| &\leq 2.
\end{aligned}
\tag{7.6}
$$

(I) We now prove claim (I) of the lemma. It is sufficient to prove that $|U| > 0$, because all faces of C are dual cells if and only if $|U| = 0$.

Firstly, observe that both N_1 and N_2 are positive. If, say, $N_1 = 0$, then there are no other vertices in chain K_1 except those of chain B. However, K_1 contains a cell D^3. We have $\mathrm{Vert}(D^3) \subset \mathrm{Vert}(B) = \mathrm{Vert}(C)$, so by lemma 18 we have $D^3 = C$, which is a contradiction since C is a prism and the tiling is 3-irreducible.

Now suppose that $|U| = 0$. If D^4 is asymmetric, then we have $|U| = N_1N_2 = 0$, so either $N_1 = 0$ or $N_2 = 0$ which is impossible by the argument above.

If D^4 is centrally symmetric, then $|U| = 0$ yields

$$
N_1N_2 = |V| \tag{7.7}
$$

This means that each vertex from $\mathrm{Vert}(K_1 \setminus B)$ is diametrically opposite to each vertex from $\mathrm{Vert}(K_1 \setminus B)$. But one vertex cannot be diametrically opposite to only one vertex. Therefore: either $N_1 = 0$, or $N_2 = 0$, or $N_1 = N_2 = 1$. In the latter case, the two vertices, one in $\mathrm{Vert}(K_1 \setminus B)$ the other in $\mathrm{Vert}(K_2 \setminus B)$, are diametrically opposite. The central symmetry of D^4 then maps vertices of $\mathrm{Vert}(B) = \mathrm{Vert}(C)$ onto themselves. That is a contradiction because C is not centrally symmetric. We have proved that $|U| > 0$, therefore not all parallelogram faces of C are dual cells.

(II) Now we prove the second claim of the lemma. By the argument above, we have $N_1, N_2 > 0$, $|U| > 0$.

From equation 7.6, it follows that the following combinations of (N_1, N_2) are possible: $(1, 1)$, $(1, 2)$, $(2, 2)$ (interchanging N_1 and N_2 if needed).

Suppose that $N_1 = N_2 = 1$. Then $|\operatorname{Vert}(D^4)| = 8$. If $\dim(D^4) = 3$, then by lemma 9 (page 54) D^4 has to be a parallelepiped, but then its combinatorial dimension is 3 which is a contradiction. Therefore $\dim(D^4) = 4$. The only possible value of $|U|$ is $|U| = 1$. It follows that D^4 is a bipyramid over a triangular prism, affinely equivalent to the polytope given in equation 7.1. In the matrix, the middle 6 columns correspond to the vertices of the prism C: columns 2-4 and 5-7 give the vertices of triangular faces of C.

We have established D^4 up to affine equivalence. Now we look at the subcell structure of D^4. In complex K, each parallelogram face of C is the base of two pyramids, with vertices $y_1 \in \operatorname{Vert}(K_1 \setminus B)$ and $\operatorname{Vert}(K_1 \setminus B)$. All 3-cells in K are the 6 pyramids, therefore K has exactly 3 parallelogram cells, faces of C. Two of these faces are dual cells, one is inscribed in an octahedral dual cell. It follows that D^4 contains 2 parallelogram dual cells.

Suppose that $N_1 = 1$, $N_2 = 2$. Then $|\operatorname{Vert}(D^4)| = 9$. By lemma 9, $\dim(D^4) = 4$. Also, D^4 cannot be centrally symmetric because it has an odd number of vertices, so there are no diametrically opposite pairs of vertices and $|U| = 2$. This, again, establishes D^4 up to affine equivalence:

$$D^4 = \text{conv}\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \tag{7.8}$$

Columns 2-7 of the matrix give the vertices of C, $\text{Vert}(K_1 \setminus B) = \{[0,0,0,0]\}$, $\text{Vert}(K_2 \setminus B) = \{[1,1,1,1],[1,1,1,0]\}$.

Dual cell D^4 has a prism face C' defined by hyperplane $x_2 = 1$. By lemma 7, all parallelogram faces of C' are subcells, which is impossible by claim (I) of the lemma.

Finally, let $N_1 = N_2 = 2$. We have from equation 7.6: $|U| \geq 2$. On the other hand, $|U| \leq 2$ since there are at most two parallelogram faces of C which extend to octahedra. Therefore $|U| = |V| = 2$. Thus D^4 is centrally symmetric, and

$$\begin{aligned} V &= \{(x, *(x)), (y, *(y))\}, \\ U &= \{(x, *(y)), (y, *(x))\} \end{aligned} \tag{7.9}$$

Since $*$ is a central symmetry, line segments $[x, *(y)]$ and $[y, *(x)]$ are translates of each other, however they are diagonals of two octahedra which are not translates of each other. This contradiction completes the proof of the lemma. □

7.2 The four cases

We have an arbitrary dual 4-cell D^4 in a 3-irreducible parallelotope tiling. To complete the proof of the theorem, we need to show that the collection $\mathcal{R}$ of incoherent parallelogram subcells in D^4 is empty.

We suppose that $|\mathcal{R}| \geq 1$. In each of the following cases, which cover all possibilities, we will get a contradiction.

1. $|\mathcal{R}| = 1$

2. D^4 is asymmetric, $|\mathcal{R}| \geq 2$ and at least one pair of parallelograms in $\mathcal{R}$ is adjacent, translate, or skew.

3. D^4 is centrally symmetric and $|\mathcal{R}| \geq 2$.

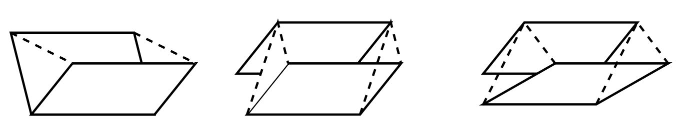
4. D^4 is asymmetric, $|\mathcal{R}| \geq 2$ and all pairs of parallelograms in $\mathcal{R}$ are complementary.

7.3 Case 1

The first case is the easiest to get rid of. We have $\mathcal{R} = \{\Pi\}$. By theorem 9 on page 38, Π is coherent: take any vertex $v \in \Pi$ and observe that all parallelogram subcells of D^4 different from Π (if any) are coherent, because they do not belong to $\mathcal{R}$. We have a contradiction.

7.4 Case 2

We now consider the case when the dual cell D^4 is asymmetric and the collection $\mathcal{R}$ of incoherent parallelogram cells contains an adjacent, translate, or skew pair of parallelograms. There is a prism C with $\mathrm{Vert}(C) \subset \mathrm{Vert}(D^4)$, and one of the parallelogram faces of C is a dual cell. The following figure shows an adjacent, translate, and a skew pair of parallelogram cells together with the prism.



Therefore, by lemma 19, D^4 is affinely equivalent to the polytope in equation 7.1 on page 76. It has exactly 2 parallelogram subcells, therefore each of these subcells is coherent in D^4, by the principle that incoherent parallelograms "come in droves", specified in theorem 9 (page 38). Therefore $\mathcal{R} = \emptyset$ which is a contradiction.

7.5 Case 3

In this case, we have a centrally symmetric dual cell D^4 in a 3-irreducible tiling. Let Π be an incoherent parallelogram subcell of D^4.

Let $*$ be the central symmetry mapping of D^4. Note that the centers of symmetry of D^4 and Π are different, because these centers are also the centers of symmetry of the corresponding $(d-4)$- and $(d-2)$-faces of the tiling. Let $C = \mathrm{conv}(\Pi \cup *(\Pi))$. We prove the following result:

D^4 is a centrally symmetric bipyramid over C and the faces of D^4 coincide with its subcells.

Once we have established this, we will apply theorem 10 on page 41 to show that all parallelogram subcells of D^4 are coherent. This will be a contradiction with the assumption that Π is incoherent.

Proof. The argument is very similar to lemma 19 on page 75, with prism substituted by the parallelepiped C.

(A). *Definition of complex K.* Let Π_k, $k = 1, \ldots, 6$ be the facets of C. We define abstract complex K as a subdivision of $\partial D^4 = \{D : D \text{ is a subcell of } D^4, D \neq D^4\}$, as follows.

Each of the vertex sets $\mathrm{Vert}(\Pi_k)$ is a centrally symmetric subset of $\mathrm{Vert}(D^4)$, hence

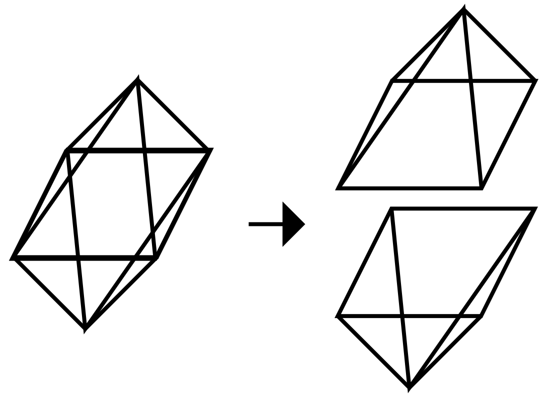


Figure 7.2: Splitting the octahedral cell

by lemma 6 there is a centrally symmetric dual subcell $E_k \subset D^4$ with $\mathrm{Vert}(\Pi_k) \subset \mathrm{Vert}(E_k)$. By lemma 4, either

1. $E_k = \Pi_k$, or

2. E_k is an octahedron (we say that Π_k is inscribed into the octahedron E_k).

If E_k is an octahedron, then we subdivide it into two pyramids with base Π_k, as shown in figure 7.2. This finishes the definition of complex K.

This is exactly the construction used in lemma 19.

(B). *Application of Alexander's separation theorem.* We define 2-chain B in the complex K as the sum of the cellular 2-chains corresponding to the facets of C. B is closed. Therefore, by the separation theorem (theorem 12 on page 78), B bounds exactly two open 3-chains in K, K_1 and K_2 where $K_1 + K_2 \equiv K \pmod 2$. Moreover, any common element of K_1 and K_2 is contained in B.

Note that the central symmetry $*$ of the dual cell D^4 maps chain K_1 onto K_2 and vice versa. It follows from the fact that mapping $*$ changes the orientation of cells in

K_1, but preserves the orientation of C.

(C). *Counting vertices.* We define set of vertex pairs U as follows.

$$U = \{(y_1, y_2) \in \mathrm{Vert}(K_1 \setminus B) \times \mathrm{Vert}(K_2 \setminus B) : y_1 \neq *(y_2)\} \tag{7.10}$$

where $*$ is the central symmetry of D^4.

The following relationship establishes a 1-1 correspondence between set U and the collection of those parallelograms Π_k which are inscribed into octahedra E_k:

$$\begin{gathered}(y_1, y_2) \to \Pi_k \\ \text{if and only if} \\ E_k = \mathrm{conv}(\Pi_k \cup \{y_1, y_2\}).\end{gathered} \tag{7.11}$$

The proof is identical to the one in lemma 19. By construction, at least two parallelograms Π_k are dual cells, therefore $|U| \leq 4$.

As we noted in paragraph (B), $*$ maps the chains K_1 and K_2 onto each other, therefore each vertex in $\mathrm{Vert}(K_1 \setminus B)$ has its centrally symmetric counterpart in $\mathrm{Vert}(K_2 \setminus B)$, and vice versa. Therefore $|\mathrm{Vert}(K_1 \setminus B)| = |\mathrm{Vert}(K_2 \setminus B)| = N$, and we have

$$|U| = N^2 - N. \tag{7.12}$$

Since $|U| \leq 4$, we can only have $N = 0, 1, 2$. We now consider each of these outcomes separately.

If $N = 0$, then $D^4 = C$, so by lemma 15 (page 58), the combinatorial dimension of D^4 is 3 which is a contradiction.

If $N = 1$ or 2, then $\dim(D^4) = 4$ (if $\dim(D^4) = 3$, then D^4 is not skinny by lemma 12 (page 56) applied to C).

If $N = 1$, then D^4 is a bipyramid over the parallelepiped C. Each parallelogram face of C belongs to two pyramids. There are 12 pyramids, 2 for each of the 6 parallelograms. All pyramids are facets of D^4. But there are a total of 12 facets. It follows that subcells of D^4 coincide with its faces.

If $N = 2$, then $|U| = 2$ and there are two parallelogram faces of C which extend to octahedra. Since D^4 is centrally symmetric, the two octahedra are mapped onto each other by the central symmetry. Therefore D^4 is a direct Minkowski sum of an octahedron and a line segment. It has a prism face C'; by lemma 7 on page 52, all facets of C' are dual cells, which is impossible by lemma 19 on page 75. □

We have thus established that D^4 is a centrally symmetric bipyramid over a parallelepiped, therefore all parallelogram subcells of D^4 are coherent by theorem 10 on page 41 and $\mathcal{R} = \emptyset$. This completes the analysis of case 3.

7.6 Case 4

This is the final and the most complex part of the proof. We have an asymmetric dual cell D^4 and the following two conditions hold for the collection $\mathcal{R}$ of incoherent parallelogram subcells of D^4:

1. each two parallelograms in $\mathcal{R}$ intersect over a vertex.

2. each vertex of a parallelogram in $\mathcal{R}$ belongs to at least one other parallelogram in $\mathcal{R}$.

We prove that $\mathcal{R} = \emptyset$ by assuming it is not true, performing a combinatorial analysis of the collection of parallelograms $\mathcal{R}$ and obtaining a contradiction.

The argument consists of three parts. Firstly, we forget about the fact that elements of $\mathcal{R}$ are parallelograms. We look at $\mathcal{R}$ as a *hypergraph*, or a system of 4-point subsets $\mathrm{Vert}(\Pi)$ of the vertex set $\mathcal{V} = \cup_{\Pi \in \mathcal{R}} \mathrm{Vert}(\Pi)$. These 4-point subsets are called *hyperedges.*

Hypergraphs can be drawn like usual graphs, with lines representing hyperedges.

We prove that $\mathcal{R}$ contains either a 5-10, or a 6-11 subsystem of parallelograms (see definition in the statement of lemma 20 below).

In the second and the third parts, we study the 5-10 and 6-11 subsystems. We remember that elements of $\mathcal{R}$ are parallelogram dual cells and use geometric properties of dual cells, established in chapter 6, to complete the analysis.

Combinatorial analysis A hypergraph with 4-element hyperedges will be called *closed* if it satisfies the following conditions:

1. each two hyperedges intersect in exactly one vertex,

2. each vertex belongs to at least two hyperedges.

Viewed as a hypergraph, the system of parallelograms $\mathcal{R}$ is closed. [1]

Let $\mathcal{R}$ be an arbitrary closed hypergraph. Let $\mathcal{V}$ be the set of vertices of $\mathcal{R}$, $V = |\mathcal{V}|$, $R = |\mathcal{R}|$. For a vertex $v \in \mathcal{V}$, its *degree* m_v is the number of hyperedges containing it.

[1] David Gregory [Gre04] suggested that the hypergraph terminology can be used in this work. Participants of the Discrete Mathematics Seminar at Queen's University and Royal Military College let us know that the dual hypergraph to a "closed" hypergraph is an example of a 'Pairwise Balanced Design' with parameter $\lambda = 1$. Special thanks to Reza Naserasr of Royal Military College, Kingston, Ontario, Canada for interest in the work.

Proposition 2 *For all $v \in \mathcal{V}$ we have $2 \leq m_v \leq 4$.*

Proof. The first inequality directly follows from the definition of a closed system. To prove the second, suppose that there are 5 or more hyperedges containing v. There is a hyperedge $\Pi \in \mathcal{R}$ such that $v \notin \Pi$, and it intersects each of the hyperedges which contain v, yet it has only 4 vertices which is a contradiction. $\square$

Calculation 1 *The following formulas hold:*

$$\sum_{v \in V} m_v = 4R, \tag{7.13}$$

$$\sum_{v \in V} m_v^2 = R(R+3), \tag{7.14}$$

$$\sum_{v \in V} (m_v - 2) = 4R - 2V, \tag{7.15}$$

$$\sum_{v \in V} (m_v - 2)^2 = R(R-13) + 4V, \tag{7.16}$$

$$|\{v \in \mathcal{V} : m_v = 4\}| = \frac{R(R-17)}{2} + 3V. \tag{7.17}$$

Proof. The first equality is obvious. The second one follows from it and from the fact that each two hyperedges have exactly one common vertex so

$$\frac{R(R-1)}{2} = \sum_{v \in V} \frac{m_v(m_v - 1)}{2}. \tag{7.18}$$

The third and fourth statements are implied from the first two. The fifth follows from the equation

$$|\{v \in \mathcal{V} : m_v = 4\}| = \frac{\sum_{v \in V} (m_v - 2)^2 - \sum_{v \in V} (m_v - 2)}{2}. \tag{7.19}$$

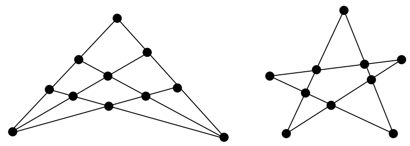


Figure 7.3: The 6-11 and 5-10 hypergraphs

which is true because $2 \leq m_v \leq 4$ and the right hand side is the sum of the indicator function $\frac{1}{2}((m_v - 2)^2 - (m_v - 2))$ of the subset of vertices v with $m_v = 4$. $\square$

For example, it follows from the calculations that if for all $v \in V$ $m_v = 2$, then $R = 5$ and $V = 10$.

Corollary 7 *$R \leq V \leq 2R$.*

Proof. The result follows from formula 7.15 and inequality $2 \leq m_v \leq 4$. $\square$

Lemma 20 *A nonempty closed hypergraph contains a subgraph isomorphic to the 5-10 hypergraph or the 6-11 hypergraph. The two hypergraphs are shown in figure 7.3.*

The hypergraphs are called so because the first has 5 hyperedges and 10 vertices, the second 6 hyperedges and 11 vertices.

Proof. We assume that closed hypergraph $\mathcal{R}$ is minimal (that is, it does not contain a proper closed subgraph).

We continue to use the notations $R = |\mathcal{R}|$, $V = |\mathcal{V}|$. Note that $R \geq 5$: since there is at least one hyperedge Q, there should be 4 more, each containing one vertex of Q. The following observations will be used in the argument below.

(A) If there are at least two vertices v_1, v_2 of degree 4 in $\mathcal{R}$, then $\mathcal{R}$ contains a 6-11 subgraph. Indeed, we can pick up 3 hyperedges containing v_1 and 3 hyperedges containing v_2, which can be done since there can be at most one hyperedge containing both vertices.

(B) If there are two vertices v_1, v_2 of degrees at least 3 which do not belong to the same hyperedge, then the system $\mathcal{R}$ contains a 6-11 subgraph. Again one picks up 3 hyperedges containing v_1 and 3 hyperedges containing v_2.

(C) If all vertices have degree 2, then $\mathcal{R}$ is a 5-10 hypergraph. This follows from the formulas on page 89.

(D) If none of (A),(B),(C) hold and $R \geq 9$, then the $\mathcal{R}$ contains a 5-10 subgraph. Let v be the vertex of degree 4, if such exists, or a vertex of degree 3 otherwise. Let $\Pi_1, \ldots, \Pi_4$ (or $\Pi_1, \ldots, \Pi_3$) be all hyperedges which contain v. Then, since (B) does not hold, all vertices of degree greater than 2 are contained in the union of hyperedges Π_i. Since (A) does not hold, all vertices in the union of hyperedges Π_i are of degree at most 3 (except, maybe, v). Let $Q_1, \ldots, Q_5 \in \mathcal{R}$ be some hyperedges different from Π_i. They form a 5-10 hypergraph (for no three of them can intersect in one point).

We are left with cases where $5 \leq R \leq 8$. We will use inequalities $R \leq V \leq 2R$, proved in corollary 7. The table below shows the number of vertices of degree 4 for each pair of values R and V, based on formula (7.17). Dash "-" means that formula 7.17 returns a negative number which is a contradiction, so corresponding pair of numbers R, V is not possible. Star "*" means that condition $R \leq V \leq 2R$ is violated.

$R \setminus V$	5..9	10	11	12	13	14	15	16
5	$-$	0	$*$	$*$	$*$	$*$	$*$	$*$
6	$-$	$-$	0	3	$*$	$*$	$*$	$*$
7	$-$	$-$	$-$	1	4	7	$*$	$*$
8	$-$	$-$	$-$	0	3	6	9	12

Table 7.1: Number of degree 4 vertices by values of V and R

Pairs of R and V where there are more than one degree 4 vertex are covered by case (A). We have to consider the rest.

If $R = 5$ and $V = 10$, $\mathcal{R}$ is a 5-10 hypergraph.

If $R = 6$ and $V = 11$, there are no degree 4 vertices and 2 degree 3 vertices v_1, v_2 because $\sum_{v \in V}(m_v - 2) = 2$ and $\sum_{v \in V}(m_v - 2)^2 = 2$ (we use the calculation above). If v_1, v_2 do not belong to the same hyperedge, then we use case (B). If they belong to some hyperedge Q, then it is sufficient to pick Q and hyperedges $Q_1, \ldots, Q_4$ at each of the vertices of Q to get a 5-10 hypergraph.

If $R = 7$ and $V = 12$, then there is 1 vertex of degree 4, but this implies that there are at least 13 vertices - a contradiction.

If $R = 8$ and $V = 12$, then $\sum_{v \in V}(m_v - 2) = 8$ and $\sum_{v \in V}(m_v - 2)^2 = 8$, so there are 8 vertices of degree 3 and 4 vertices of degree 2. Take a vertex v of degree 3 and let Q_1, Q_2 and Q_3 be all hyperedges containing it. If there is another vertex u so that $m_u = 3$ and $u \notin \cup_{i=1}^{3} Q_i$, then we use case (B). Otherwise the remaining 7 vertices of degree 3 are found among vertices of Q_i, $i = 1 \ldots 3$, which means that on one of Q_i all vertices are of degree 3. Therefore $\mathcal{R} \setminus \{Q_i\}$ is closed so $\mathcal{R}$ is not minimal contrary to the assumption. The contradiction finalizes the proof. $\square$

We consider the cases when $\mathcal{R}$ contains a 5-10 subgraph and when $\mathcal{R}$ contains a 6-11 subgraph separately.

Hypergraph $\mathcal{R}$ contains a 5-10 subgraph $\mathcal{R}'$

We calculate all possible systems $\mathcal{R}'$ and then use geometric properties of dual cells to obtain the desired contradiction.

So far our analysis only considered parallelograms as hyperedges, unordered 4-element subsets of $\mathcal{V}$. Now we take into account the fact that parallelograms are centrally symmetric 4-vertex polygons. We match diagonally opposite vertices of each parallelogram. As a result, the vertices of each hyperedge are split into two pairs. We will call this construction "vertex matching".

We will now classify all possible vertex matchings of the 5-10 hypergraph, up to isomorphism.

It is convenient to use the dual graph to the 5-10 hypergraph, the complete graph on 5 elements K_5. Matching of vertices in each hyperedge of $\mathcal{R}'$ corresponds to matching of edges of K_5 incident to the the same vertex.

Matching of edges of K_5 at each vertex is equivalent to building a snow ploughing scheme on "roads"-edges of K_5, where multiple ploughing machines travel on circuits so that each edge is cleaned, and no edge is cleaned twice [2]. Since the thesis was written in Canada, this analogy makes a lot of sense to us.

Correspondence between snow ploughing schemes and edge matchings is straightforward: as each machine travels on the graph, we simply match the entering and exiting roads (edges) at each crossing (vertex). Conversely, matching of edges at each vertex tells a machine which way to go from a crossing (vertex) after entering from a given road (edge).

We will write a snow ploughing scheme as a system of vertex cycles, indicating vertices that each ploughing machine visits.

Lemma 21 *Vertices of K_5 can be labeled by $1, \ldots, 5$ so that a snow ploughing scheme of K_5 is one of the following:*

1. $(1, 2, 3, 1, 4, 2, 5, 4, 3, 5)$
2. $(1, 3, 5, 1, 2, 5, 4, 2, 3, 4)$

[2] A circuit that passes each edge exactly once is called *Eulerian cycle*.

3. $(1,2,3,1,4,2,5,3,4,5)$

4. $(1,2,3,1,4,5,2,4,3,5)$

5. $(1,4,2,5,3,4,5)$, $(1,2,3)$

6. $(1,2,3,4)$, $(1,5,2,4,5,3)$

7. $(1,2,3,4,5)$, $(1,3,5,2,4)$

8. $(1,5,2,4)$, $(1,2,3)$, $(3,4,5)$

Proof. We used a computer program to find all Eulerian cycles on K_5 (lines 1-4). For a scheme of two or more cycles (lines 5-8), an easy manual classification proves the result. $\square$

From this lemma we know all vertex matchings on edges of a 5-10 hypergraph, up to isomorphism. We denote the hyperedges/parallelograms in $\mathcal{R}'$ by $\Pi_1, \ldots, \Pi_5$, and label vertices of the hypergraph $\mathcal{R}'$ by v_{ij}, $i \neq j$, where v_{ij} stands for the intersection of hyperedges Π_i and Π_j. Now, each vertex is a point in space, and matching pairs of vertices of the same hyperedge represent diagonals of a parallelogram. This allows us to write a system of linear equations.

For example, snow ploughing scheme 1 translates into the following system:

$$\begin{aligned}
v_{12} + v_{15} &= v_{13} + v_{14}, \\
v_{12} + v_{23} &= v_{24} + v_{25}, \\
v_{23} + v_{13} &= v_{34} + v_{35}, \\
v_{14} + v_{24} &= v_{34} + v_{45}, \\
v_{25} + v_{45} &= v_{15} + v_{35}
\end{aligned} \tag{7.20}$$

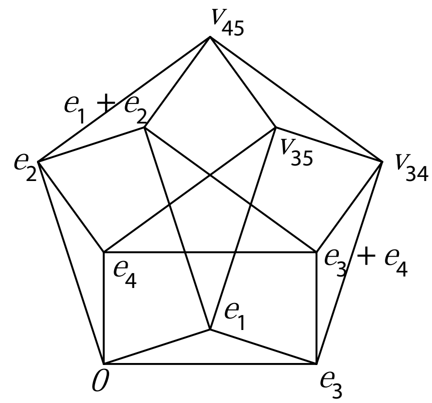


Figure 7.4: System of parallelograms from line 2 of table 7.21

We choose a coordinate system $e_1, \ldots, e_d$ in $\mathbb{R}^d$ so that parallelograms Π_1 and Π_2 are $\mathrm{conv}\{0, e_1, e_2, e_1+e_2\}$, $\mathrm{conv}\{0, e_3, e_4, e_3+e_4\}$. This is possible by the assumption that each two parallelograms are complementary, that is, they share a vertex and span a 4-space.

We then consider each of the cases in lemma 21, write down the corresponding linear equations and solve them. The results, easily verifiable, are shown in table 7.21. Each solution is shown in the format $[v_{12}, v_{13}, v_{14}, v_{15}, v_{23}, v_{24}, v_{25}, v_{34}, v_{35}, v_{45}]$. For each solution, we indicate if it is immediately obvious that the parallelogram system cannot occur in a dual 4-cell, in column "reason for contradiction".

We are left to prove that systems of parallelograms with vertices shown in lines 2 and 7 of table 7.21 cannot occur in the dual 4-cell.

First consider row 7. We have

$$[v_{12}, v_{35}, v_{15}, v_{23}, v_{34}, v_{45}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0\ldots & & & & & \end{bmatrix}. \tag{7.22}$$

Case	Solution	Reason for contradiction
1	None	n/a
2	$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & a^1-1 & a^1 & a^1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & a^2-1 & a^2-1 & a^2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & a^3+1 & a^3 & a^3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & a^4+1 & a^4+1 & a^4 \\ 0 & \ldots & & & & & & a^5 & a^5 & a^5 \\ \ldots \end{bmatrix}$	See below
3	None	n/a
4	$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 \\ \ldots \end{bmatrix}$	$v_{34} = v_{15}$
5	None	n/a
6	$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 \\ \ldots \end{bmatrix}$	$v_{34} = v_{12}$
7	$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 1 \\ 0 \\ \ldots \end{bmatrix}$	See below
8	$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 \\ \ldots \end{bmatrix}$	$v_{45} = v_{12}$

(7.21)

Table 7.2: All parallelogram systems with 5-10 hypergraph, up to affine equivalence

The 6 points $v_{12}, v_{35}, v_{15}, v_{23}, v_{34}, v_{45}$ form a centrally symmetric subset of $\text{Vert}(D^4)$ (the center of symmetry is $[0, 1/2, 1/2, 0]$). By lemma 6, there is a centrally symmetric dual cell $D \subset D^4$ with $\{v_{12}, v_{35}, v_{15}, v_{23}, v_{34}, v_{45}\} \subset \text{Vert}(D)$. Since D^4 is asymmetric, D is a proper subcell of D^4 so $\text{combdim}(D) \le 3$. But $\text{combdim}(D) \ge 3$ since the 6 points in equation 7.22 span a 3-dimensional space. Therefore $\text{combdim}(D) = 3$. By lemma 3, D is a simplex, an octahedron or a pyramid. But D is centrally symmetric so D is an octahedron. Line segment $[v_{15}, v_{45}]$ is an edge of the octahedron, so it is a dual 1-cell by lemma 3. However, $[v_{15}, v_{45}]$ is a diagonal of parallelogram cell Π_5, which is impossible.

Next consider the system of parallelograms in row 2 of table 7.21 (shown schematically in figure 7.4). This is the hardest case of all, because there is a vector parameter $a \in \mathbb{R}^d$ in the solution.

(A) *Setup.* We introduce the following notations. Firstly, the new origin in the coordinate system will be $E = e_1 + e_4 = v_{25} + v_{13}$.

$$\begin{aligned} y_1 &= v_{52} - E, \\ y_2 &= v_{13} - E, \\ y_3 &= v_{24} - E, \\ y_4 &= v_{35} - E, \\ y_5 &= v_{41} - E. \end{aligned}$$

We then have:

$$\begin{aligned} E + y_1 + y_2 &= v_{12}, \\ E + y_2 + y_3 &= v_{23}, \\ E + y_3 + y_4 &= v_{34}, \\ E + y_4 + y_5 &= v_{45}, \\ E + u_5 + u_1 &= v_{51}. \end{aligned}$$

From this moment on, we shift the coordinate system, so that $E = 0$. We then have

$$\begin{aligned}
\Pi_1 &= \operatorname{conv}\{y_5, y_2, y_5 + y_1, y_1 + y_2\},\\
\Pi_2 &= \operatorname{conv}\{y_1, y_3, y_1 + y_2, y_2 + y_3\},\\
\Pi_3 &= \operatorname{conv}\{y_2, y_4, y_2 + y_3, y_3 + y_4\},\\
\Pi_4 &= \operatorname{conv}\{y_3, y_5, y_3 + y_4, y_4 + y_5\},\\
\Pi_5 &= \operatorname{conv}\{y_4, y_1, y_4 + y_5, y_5 + y_1\}
\end{aligned}$$

By theorem 11, statement 3 (page 63) the linear space $L^4 =< y_1, y_2, y_3, y_4 >= \operatorname{aff}(\Pi_2 \cup \Pi_3)$ is complementary to $\operatorname{lin}(F^{d-4} - F^{d-4})$, where F^{d-4} is the face of the tiling corresponding to the dual cell D^4. Let h be the projection along F^{d-4} onto L^4. By the same theorem, each two parallelograms $h(\Pi_i)$ and $h(\Pi_j)$ have a common vertex and span a 4-space.

Now we will lift the system of parallelograms $h(\Pi_1), \ldots, h(\Pi_5)$ into a 5-dimensional space. Take a vector $x \notin L^4$, and let $u_i = h(y_i) = y_i$, $i = 1, \ldots, 4$, $u_5 = h(y_5) + x$.

We have obtained a pleasantly symmetric representation of the lifted system of parallelograms, with vertex set $\{u_1, \ldots, u_5, u_1{+}u_2, \ldots, u_5{+}u_1\}$. This system of parallelograms is invariant under transformations of space induced by cyclical substitution of indices. The lifted parallelograms are:

$$\begin{aligned}
P_1 &= \operatorname{conv}\{u_5, u_2, u_5 + u_1, u_1 + y_2\},\\
P_2 &= \operatorname{conv}\{u_1, u_3, u_1 + u_2, u_2 + y_3\},\\
P_3 &= \operatorname{conv}\{u_2, u_4, u_2 + u_3, u_3 + y_4\},\\
P_4 &= \operatorname{conv}\{u_3, u_5, u_3 + u_4, u_4 + y_5\},\\
P_5 &= \operatorname{conv}\{u_4, u_1, u_4 + u_5, u_5 + y_1\}
\end{aligned}$$

Let $Q = \operatorname{conv}(\{u_1, \ldots, u_5, u_1{+}u_2, \ldots, u_5{+}u_1\})$. Note that parallelograms $P_1, \ldots, P_5$ are faces of Q (a trivial computation proves it). Let p be the projection from $< u_1, \ldots, u_5 >$ to $< u_1, \ldots, u_4 >=< y_1, \ldots, y_4 >$ along vector x.

We have $p(Q) \subset h(D^4)$. We will analyze vectors $x = (x^1, \ldots, x^5)$ where x^i are its coordinates in the basis $u_1, \ldots, u_5$. Our objective is to test all vectors $x \neq 0$, to prove that inclusion $p(Q) \subset h(D^4)$ is impossible.

(B) *Tests for vector x.* We now present two conditions that vector x must satisfy.

The first condition arises from theorem 11, statement 3 (page 63). It says that each two parallelograms $h(\Pi_i) = p(P_i)$, $h(\Pi_j) = p(P_j)$ span a 4-space. The null space $< x >$ of projection p must be therefore complementary to the space spanned by each pair of parallelograms P_i, P_j. It follows that

$$\begin{gathered} x^i \neq 0, \\ i = 1, \dots, 5, \end{gathered} \tag{7.23}$$

The second condition follows from statements 2 and 3 of lemma 4. Let v be an arbitrary vertex of Q, and suppose that $v \notin P_i$. Let C_v be the cone bounded by the hyperplanes which define facets of $Q - v$ containing 0. In other words, $C_v = \text{cone}(Q - v)$. Polytope $Q - v$ coincides with C_v in some neighborhood $U_\epsilon(0)$ of 0. Then:

$$x \notin \text{relint}(C_v - \text{lin}(P_i - P_i)) \cup -\text{relint}(C_v - \text{lin}(P_i - P_i)) \tag{7.24}$$

We call this condition *testing parallelogram P_i against vertex v.*

To show that the condition holds, suppose eg. $x \in \text{relint}(C_v - \text{lin}(P_i - P_i))$. Then $x = z_1 - z_2$, where $z_1 \in \text{relint}(C_v)$, $z_2 \in \text{lin}(P_i - P_i)$. We have $z_1 \neq 0$, since C_v is a cone with vertex 0 and $0 \notin \text{relint}(C_v)$.

Consider the case when $z_2 = 0$. Then $x = z_1 \in \text{relint}(C_v)$. But $< x >$ is the null space of projection p, therefore $p(v)$ is not a vertex of $p(Q)$, which is a contradiction: both polytopes $p(Q)$ and Q have 10 vertices.

Now suppose that $z_2 \neq 0$. We then have

$$z_2 = \alpha_1 t_1 + \alpha t_2 \tag{7.25}$$

where t_1, t_2 are edge vectors of parallelogram P_i with appropriately chosen directions, $\alpha_1, \alpha_2 \geq 0$. Since x is interesting to us only as the direction vector for linear projection p, we can multiply vector x and vectors z_1 and z_2 by the same small positive number so that $z_1 \in U_\epsilon(0)$, $\alpha_1, \alpha_2 < 1$. We then have

$$\begin{aligned} z_1 &\in \operatorname{relint}(Q - v), \\ z_2 &\in P_i - v' \end{aligned} \tag{7.26}$$

for some choice of vertex v' of P_i. From $x = z_1 - z_2$, we have $p(z_1) = p(z_2)$. It follows that:

$$\begin{aligned} \operatorname{relint}(p(Q - v)) \cap p(P_i - v') &\neq \emptyset, \\ \operatorname{relint}(p(Q - v)) \cap p(Q - v') &\neq \emptyset, \\ \operatorname{relint}(p(Q) - (p(v) - p(v'))) \cap \operatorname{relint}(p(Q)) &\neq \emptyset, \\ \operatorname{relint}(h(D^4) - (p(v) - p(v'))) \cap \operatorname{relint}(h(D^4)) &\neq \emptyset. \end{aligned} \tag{7.27}$$

By the choice of v and P_i, points $p(v)$ and $p(v')$ are different vertices of $h(D^4)$, therefore polytope $h(D^4)$ is not skinny. This is a contradiction with lemma 8 on page 53. We have proved that condition 99 is a valid test for vector x.

Another simple yet useful idea is that the polytope Q is invariant under a group consisting of 5 cyclical coordinate substitutions. This means that whatever condition on the projection vector x we have derived, up to 5 more conditions can be produced by cyclically substituting indices.

(C) *Application of the tests.* Now we test vector x against the conditions specified above. We used computer software PORTA by T. Christof (University of Heidelberg) to perform polytope computations.

First, we test parallelogram P_2 against each of the vertices u_2, u_4, u_5. We conclude that the projection vector x does not belong to any of the following open cones and their centrally symmetric images:

$$\begin{gathered}\{x : x^4 < 0, x^5 > 0, x^1 + x^3 > 0\},\\ \{x : x^4 > 0, x^5 < 0, x^1 + x^3 > 0\},\\ \{x : x^4 > 0, x^5 > 0, x^1 + x^3 > 0\}\end{gathered} \tag{7.28}$$

We also know that $x^1 \neq 0, \ldots, x^5 \neq 0$ because the projection vector x must be complementary to the affine 4-space spanned by any pair of parallelograms P_i, P_j. Therefore x must satisfy one of the following conditions:

$$\begin{aligned}x \in K^1_+ &= \{x : x^4 > 0, x^5 > 0, x^1 + x^3 < 0\},\\ x \in K^1_- &= \{x : x^4 < 0, x^5 < 0, x^1 + x^3 > 0\},\\ x \in K^1_0 &= \{x : x^1 + x^3 = 0\}\end{aligned} \tag{7.29}$$

We then rotate the indices in the inequalities to obtain sets K^i_+, K^i_-, K^i_0 for $i = 1, \ldots, 5$. For example, $K^2_0 = \{x : x^2 + x^4 = 0\}$. We get

$$\begin{aligned}x \in \cap_{i=1}^5 (K^i_+ \cup K^i_- \cup K^i_0),\\ i = 1, \ldots, 5.\end{aligned} \tag{7.30}$$

Opening brackets in the expression, we have

$$x \in \bigcup_{[\sigma_1,\ldots,\sigma_5],\sigma_i\in\{+,-,0\}} (K^1_{\sigma_1} \cap K^2_{\sigma_2} \cap K^3_{\sigma_3} \cap K^4_{\sigma_4} \cap K^5_{\sigma_5}). \tag{7.31}$$

Now we test parallelogram P_2 against the vertex $u_4 + u_5$. The result is:

$$\begin{aligned} x \notin J_+ = \{x^5 < 0, x^4 < 0, x^1 + x^3 + x^4 + x^5 < 0, x^1 + x^3 > 0\}&, \\ \text{and}& \\ x \notin J_- = \{x^5 > 0, x^4 > 0, x^1 + x^3 + x^4 + x^5 > 0, x^1 + x^3 < 0\}&. \end{aligned} \tag{7.32}$$

Using cyclical substitutions, we obtain 10 conditions from this test.

After a trivial calculation performed with the PORTA software, the only vector which passes the tests is $x = [-1, -1, -1, 1, 1]$ (and its multiples). It is treated in a different way.

(D) *Case of* $x = [-1, -1, -1, 1, 1]$. The polytope $p(Q)$ is the projection of Q along x onto the space spanned by $u_1, \ldots, u_4$. Polytope $p(Q)$ has the following vertices (coordinates are shown in the basis $u_1, \ldots, u_4$):

$$\begin{aligned} p(u_1) &= h(y_1) = [1, 0, 0, 0], \\ p(u_2) &= h(y_2) = [0, 1, 0, 0], \\ p(u_3) &= h(y_3) = [0, 0, 1, 0], \\ p(u_4) &= h(y_4) = [0, 0, 0, 1], \\ p(u_5) &= h(y_5) = [1, 1, 1, -1], \\ p(u_1 + u_2) &= h(y_1 + y_2) = [1, 1, 0, 0], \\ p(u_2 + u_3) &= h(y_2 + y_3) = [0, 1, 1, 0], \\ p(u_3 + u_4) &= h(y_3 + y_4) = [0, 0, 1, 1], \\ p(u_4 + u_5) &= h(y_4 + y_5) = [1, 1, 1, 0], \\ p(u_5 + u_1) &= h(y_5 + y_1) = [2, 1, 1, -1] \end{aligned} \tag{7.33}$$

Consider the line segment $[[1,1,1,0],[1,\frac{1}{2},1,0]]$. The following inclusions hold:

$$\begin{aligned} [[1,1,1,0],[1,\frac{1}{2},1,0]] &\subset p(Q), \\ [[1,1,1,0],[1,\frac{1}{2},1,0]] &\subset p(Q)+p(u_4+u_5-u_1-u_2) \end{aligned} \tag{7.34}$$

Let $t = y_4 + y_5 - y_1 - y_2$. Since $p(Q) \subset h(D^4)$,

$$p(Q)+p(u_4+u_5-u_1-u_2) = p(Q)+h(t) \subset h(D^4)+h(t) = h(D^4+t), \tag{7.35}$$

we have

$$[[1,1,1,0],[1,\frac{1}{2},1,0]] \subset h(D^4) \cap h(D^4+t). \tag{7.36}$$

By lemma 4 on page 46, polytopes D^4 and D^4+t can be separated by a hyperplane N so that the following conditions hold:

$$\begin{aligned} D = D^4 \cap (D^4+t) = N \cap D^4 &= N \cap (D^4+t), \\ \operatorname{lin}(F^{d-4}-F^{d-4}) &\subset N-N, \\ D \text{ is a dual cell}& \end{aligned} \tag{7.37}$$

We therefore have

$$[[1,1,1,0],[1,\frac{1}{2},1,0]] \subset N \tag{7.38}$$

It follows that

$$[1,0,1,0] = h(y_1+(y_4+y_5-y_1-y_2)) \in N, \tag{7.39}$$

hence $y_1+(y_4+y_5-y_1-y_2) \in N$. Point $y_1+(y_4+y_5-y_1-y_2) = y_4+y_5-y_2$ is a vertex of $D^4+(y_4+y_5-y_1-y_2)$. By equation (7.37), it is also a vertex of D^4.

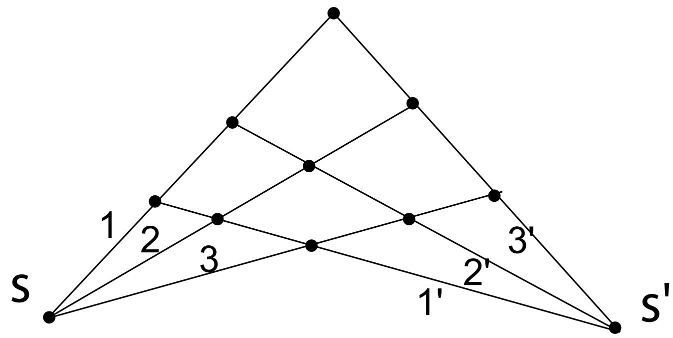


Figure 7.5: The 6-11 hypergraph

Note that $\mathrm{Vert}(\Pi_2) \cup \{y_4 + y_5, y_4 + y_5 - y_2\}$ is a vertex set of a triangular prism.

This, by lemma 19, implies that $|\,\mathrm{Vert}(D^4)| = 8$, however we know that D^4 contains 10 vertices of the 5 parallelograms $\Pi_1, \ldots, \Pi_5$. The contradiction finishes the 5-10 parallelogram system analysis.

6-11 system The 6-11 hypergraph is shown in figure 7.5.

Since each hyperedge is the quartet of vertices of a parallelogram in $\mathcal{R}'$, we can match vertices of each hyperedge into pairs corresponding to diagonals of the parallelogram. Let S, S' be the two collections of parallelograms, each sharing a common vertex s, s'. The vertex s on each of parallelograms $\Pi \in S$ is matched with a vertex $\Pi \cap \Pi'$ for some $\Pi' \in S'$. This establishes a mapping $\sigma : S \to S'$ by $\sigma(\Pi) = \Pi'$. Similarly we define a mapping $\sigma' : S' \to S$. The vertex matching is completely defined by the two mappings σ and σ'.

Lemma 22 *Parallelograms in collections S, S' can be labeled by symbols 1-3 and $1'$-$3'$ so that the mappings σ_1, σ_2 are:*

1. $\begin{pmatrix} 1 & 2 & 3 \\ 1' & 1' & 1' \end{pmatrix}, \begin{pmatrix} 1' & 2' & 3' \\ 1 & 1 & 1 \end{pmatrix}$,
2. $\begin{pmatrix} 1 & 2 & 3 \\ 1' & 1' & 1' \end{pmatrix}, \begin{pmatrix} 1' & 2' & 3' \\ 1 & 1 & 2 \end{pmatrix}$,

3. $\begin{pmatrix} 1 & 2 & 3 \\ 1' & 1' & 1' \end{pmatrix}, \begin{pmatrix} 1' & 2' & 3' \\ 1 & 2 & 2 \end{pmatrix}$,

4. $\begin{pmatrix} 1 & 2 & 3 \\ 1' & 1' & 1' \end{pmatrix}, \begin{pmatrix} 1' & 2' & 3' \\ 1 & 2 & 3 \end{pmatrix}$,

5. $\begin{pmatrix} 1 & 2 & 3 \\ 1' & 1' & 2' \end{pmatrix}, \begin{pmatrix} 1' & 2' & 3' \\ 1 & 1 & 2 \end{pmatrix}$,

6. $\begin{pmatrix} 1 & 2 & 3 \\ 1' & 2' & 1' \end{pmatrix}, \begin{pmatrix} 1' & 2' & 3' \\ 1 & 1 & 2 \end{pmatrix}$,

7. $\begin{pmatrix} 1 & 2 & 3 \\ 2' & 1' & 1' \end{pmatrix}, \begin{pmatrix} 1' & 2' & 3' \\ 1 & 1 & 2 \end{pmatrix}$,

8. $\begin{pmatrix} 1 & 2 & 3 \\ 1' & 1' & 2' \end{pmatrix}, \begin{pmatrix} 1' & 2' & 3' \\ 1 & 2 & 1 \end{pmatrix}$, *reduces to 6* ($S \leftrightarrow S'$),

9. $\begin{pmatrix} 1 & 2 & 3 \\ 1' & 2' & 1' \end{pmatrix}, \begin{pmatrix} 1' & 2' & 3' \\ 1 & 2 & 1 \end{pmatrix}$,

10. $\begin{pmatrix} 1 & 2 & 3 \\ 2' & 1' & 1' \end{pmatrix}, \begin{pmatrix} 1' & 2' & 3' \\ 1 & 2 & 1 \end{pmatrix}$,

11. $\begin{pmatrix} 1 & 2 & 3 \\ 1' & 1' & 2' \end{pmatrix}, \begin{pmatrix} 1' & 2' & 3' \\ 2 & 1 & 1 \end{pmatrix}$, *reduces to 7* ($S \leftrightarrow S'$),

12. $\begin{pmatrix} 1 & 2 & 3 \\ 1' & 2' & 1' \end{pmatrix}, \begin{pmatrix} 1' & 2' & 3' \\ 2 & 1 & 1 \end{pmatrix}$, *reduces to 10* ($S \leftrightarrow S'$),

13. $\begin{pmatrix} 1 & 2 & 3 \\ 2' & 1' & 1' \end{pmatrix}, \begin{pmatrix} 1' & 2' & 3' \\ 2 & 1 & 1 \end{pmatrix}$,

14. $\begin{pmatrix} 1 & 2 & 3 \\ 1' & 1' & 2' \end{pmatrix}, \begin{pmatrix} 1' & 2' & 3' \\ 1 & 2 & 3 \end{pmatrix}$,

15. $\begin{pmatrix} 1 & 2 & 3 \\ 1' & 2' & 1' \end{pmatrix}, \begin{pmatrix} 1' & 2' & 3' \\ 1 & 2 & 3 \end{pmatrix}$,

16. $\begin{pmatrix} 1 & 2 & 3 \\ 2' & 1' & 1' \end{pmatrix}, \begin{pmatrix} 1' & 2' & 3' \\ 1 & 2 & 3 \end{pmatrix}$,

17. $\begin{pmatrix} 1 & 2 & 3 \\ 1' & 2' & 3' \end{pmatrix}, \begin{pmatrix} 1' & 2' & 3' \\ 1 & 2 & 3 \end{pmatrix}$,

18. $\begin{pmatrix} 1 & 2 & 3 \\ 1' & 2' & 3' \end{pmatrix}, \begin{pmatrix} 1' & 2' & 3' \\ 2 & 1 & 3 \end{pmatrix}$.

Proof. The sizes of images of the mappings are independent of the labeling of parallelograms. We use them to classify the cases we need to consider:

$\lvert \mathrm{Im}(\sigma) \rvert$	$\lvert \mathrm{Im}(\sigma') \rvert$
1	1
1	2
1	3
2	2
2	3
3	3

(7.40)

We then use direct inspection. □

The common vertex of parallelograms k and l' will be denoted $v_{kl'}$. Since parallelograms 1 and 2 share exactly one common vertex, they are complementary and span a 4-space. We choose a basis $e_1, \ldots, e_d$ in $\mathbb{R}^d$ so that parallelograms 1 and 2 are $\mathrm{conv}\{0, e_1, e_2, e_1 + e_2\}$, $\mathrm{conv}\{0, e_3, e_4, e_3 + e_4\}$.

We then solve the systems of linear equations for points $v_{kl'}$ which arise from the combinatorial information that tells us which vertices of each hyperedge form a diagonal. For example, the vertex matching represented by line 1 in lemma 22 results in the following system of linear equations:

$$
\begin{aligned}
s + v_{11'} &= v_{12'} + v_{13'}, \\
s + v_{21'} &= v_{22'} + v_{23'}, \\
s + v_{31'} &= v_{32'} + v_{33'}, \\
s' + v_{11'} &= v_{21'} + v_{31'}, \\
s' + v_{12'} &= v_{22'} + v_{32'}, \\
s' + v_{13'} &= v_{23'} + v_{33'}
\end{aligned}
\tag{7.41}
$$

The solutions are presented in the tables 7.6, 7.6, and 7.6 below, in the format $[s, v_{11'}, v_{12'}, v_{13'}, v_{21'}, v_{22'}, v_{23'}, v_{31'}, v_{32'}, v_{33'}, s']$. The coordinates are in the basis $e_1, \ldots, e_d$. Column "reason for contradiction" shows why each outcome is impossible.

In each case we see that either some of the 11 vertices of the parallelogram system coincide, or are equivalent to each other modulo 2Λ where Λ is the lattice of the tiling. This completes the analysis of the 6-11 system of parallelograms, and finishes the proof of the theorem.

Case	Solution	Reason for contradiction
1	$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & -1 & 0 \\ 0\dots \end{bmatrix}$	$s = s'$
2	$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 3 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -3 & -2 & -1 & -2 \\ 0\dots \end{bmatrix}$	$s' \equiv s \pmod{2\Lambda}$
3	$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 3 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 3 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -3 & -1 & -2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -3 & -2 & -1 & -2 \\ 0\dots \end{bmatrix}$	$s' \equiv s \pmod{2\Lambda}$
4	$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -3 & 2 & -2 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 3 & -2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 2 & -1 & 2 \\ 0\dots \end{bmatrix}$	$s' \equiv s \pmod{2\Lambda}$
5	$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0\dots \end{bmatrix}$	$s' \equiv s \pmod{2\Lambda}$
6	$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 3 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 1 & 0 \\ 0\dots \end{bmatrix}$	$s' \equiv s \pmod{2\Lambda}$
7	$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -3 & -2 & -1 & -2 \\ 0\dots \end{bmatrix}$	$s' \equiv s \pmod{2\Lambda}$
8	Reduces to 6	

(7.42)

Table 7.3: All parallelogram systems with 6-11 hypergraph, up to affine equivalence

Case	Solution	Reason for contradiction
9	$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 3 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -3 & -1 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -1 & 0 \\ 0\ldots \end{bmatrix}$	$s' \equiv s \pmod{2\Lambda}$
10	$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 3 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -3 & -1 & -2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & -1 & 0 \\ 0\ldots \end{bmatrix}$	$s' \equiv s \pmod{2\Lambda}$
11	Reduces to 7	
12	Reduces to 10	
13	$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & -3 & -1 & -2 & -2 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 3 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 3 & 2 & 1 & 2 \\ 0\ldots \end{bmatrix}$	$s' \equiv s \pmod{2\Lambda}$
14	$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 3 & 1 & -2 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 3 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -3 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0\ldots \end{bmatrix}$	$s' \equiv s \pmod{2\Lambda}$
15	$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -3 & 2 & -2 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 3 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & -1 & 2 \\ 0\ldots \end{bmatrix}$	$s' \equiv s \pmod{2\Lambda}$
16	$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -3 & 2 & -2 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 3 & -2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 2 & -1 & 2 \\ 0\ldots \end{bmatrix}$	$s' \equiv s \pmod{2\Lambda}$

(7.43)

Table 7.4: All parallelogram systems with 6-11 hypergraph, up to affine equivalence (continued)

Case	Solution	Reason for contradiction
17	$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0\ldots \end{bmatrix}$	$s' \equiv s \pmod{2\Lambda}$
18	$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1/3 & 2/3 & 1/3 & 2/3 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2/3 & -1/3 & 1/3 & 2/ \\ 0\ldots \end{bmatrix}$	not in a convex position

(7.44)

Table 7.5: All parallelogram systems with 6-11 hypergraph, up to affine equivalence (continued)

Chapter 8

Venkov graphs and the reducibility of parallelotopes

A parallelotope is called *reducible* if it can be represented as a direct Minkowski sum of two parallelotopes of smaller positive dimensions. To establish the Voronoi conjecture on parallelotopes, it is sufficient to study irreducible parallelotopes.

In this chapter we present our criterion of parallelotope reducibility in terms of its combinatorial structure.

We use the material in chapters 1, 2, and the quality translation theorem (section 3.2). Otherwise this chapter is independent from the rest of the thesis.

8.1 Venkov graph

The central object is the *Venkov graph* of a parallelotope.

Definition 8 *Let $P \subset \mathbb{R}^d$ be a parallelotope, $d \geq 2$. The graph $V(P)$ with the vertex set composed of the pairs of opposite facets of P where*

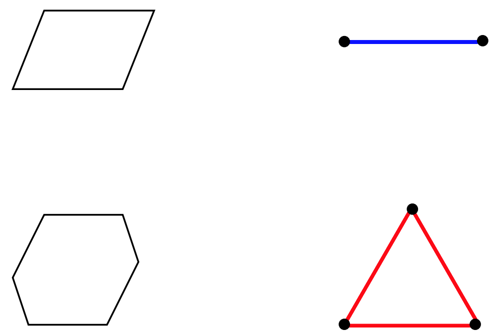


Figure 8.1: All 2-dimensional parallelotopes up to combinatorial equivalence, shown with their Venkov graphs

1. *two distinct vertices are connected by a blue edge if the corresponding pairs of facets belong to a common quadruple belt,*

2. *three distinct vertices are spanned by a red triangle if the corresponding three pairs belong to a common hexagonal belt,*

is called the Venkov graph of the parallelotope P.

Symbol $\oplus$ stands for the direct Minkowski sum. Remember that by the Venkov criterion, theorem 4 on page 8, the projection of a parallelotope along a $(d-2)$-face onto a complementary 2-plane is either a centrally symmetric hexagon or a parallelogram. The edges of the polygon are the images of facets of the parallelotope, which constitute a *belt defined by the* $(d-2)$*-face*. The belt and the $(d-2)$-face are called hexagonal or quadruple depending on whether the polygon is a hexagon or a parallelogram.

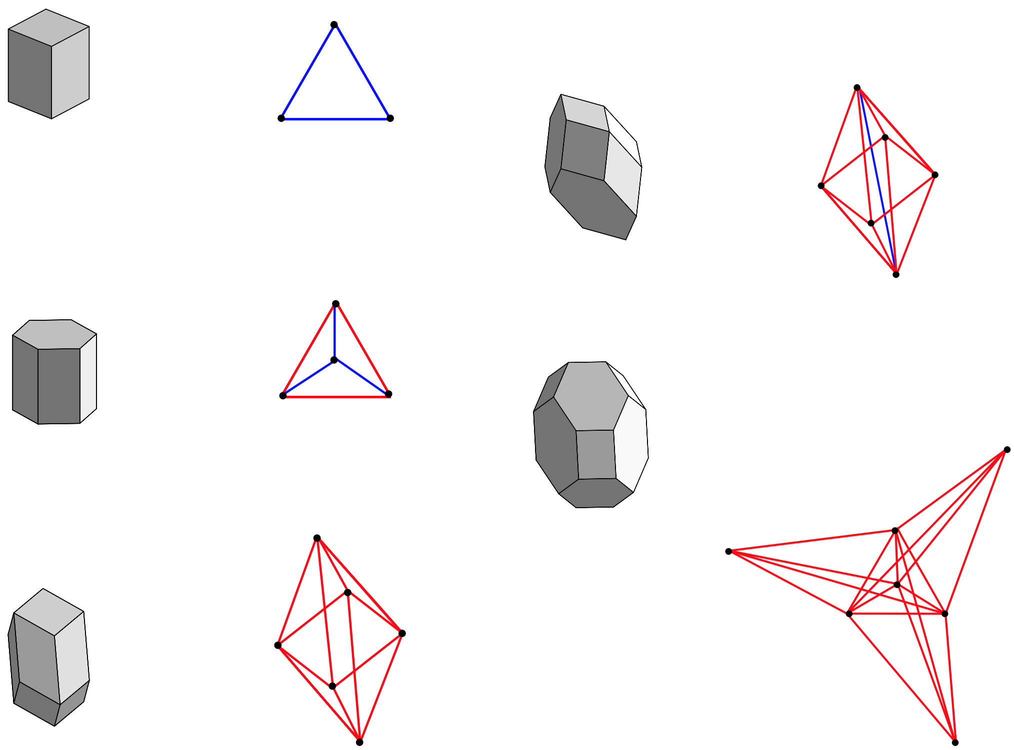


Figure 8.2: All 3-dimensional parallelotopes up to combinatorial equivalence, shown with their Venkov graphs

8.2 Reducibility criterion for parallelotopes

Theorem 13 1. *Suppose a parallelotope P is a direct Minkowski sum of parallelotopes P_1, P_2. Assume that 0 is the center of symmetry of each of the parallelotopes P, P_1, P_2. Let V_1, V_2 be the subgraphs of $V(P)$ induced by vertices of the types $\pm F_1 \oplus P_2$ and $\pm F_2 \oplus P_1$ respectively where F_1 is any facet of P_1, F_2 any facet of P_2. Then*

   (a) *The mapping from $V(P_i)$ to V_i where vertex $\pm F_i$ goes to $\pm F_i \oplus P_j$ is an isomorphism, for all $\{i, j\} = \{1, 2\}$.*

   (b) *The subgraphs V_1 and V_2 are completely cross connected by blue edges. There are no red edges between V_1 and V_2.*

2. *If the Venkov graph of P is not connected by red edges, then the parallelotope P is reducible.*

The first statement says that the Venkov graph of a parallelotope $P = P_1 \oplus P_2$ can be constructed by taking the Venkov graphs $V(P_1)$ and $V(P_2)$ and joining each vertex of the first graph with each each vertex of the second graph by a blue edge.

Together, the two statements imply our criterion of parallelotope irreducibility, stated in the introduction:

Theorem 2 *A parallelotope is irreducible if and only if its Venkov graph is connected by red edges.*

Corollary 8 *If a parallelotope is irreducible, then any two facets in the tiling can be connected by a combinatorial path on the $(d-1)$-skeleton of the tiling where the joints are hexagonal $(d-2)$-faces.*

Proof. The Venkov graph of an irreducible parallelotope is connected by red edges, therefore each two facets of a given parallelotope can be connected by a combinatorial path on the boundary of the parallelotope, where the joints are hexagonal $(d-2)$-faces. The result of the corollary follows from the fact that each pair of parallelotopes in the tiling can be connected by a combinatorial path. $\square$

The next corollary directly follows from the definition of canonical scaling on page 19.

Corollary 9 *Suppose that a parallelotope tiling has a canonical scaling. Then the following two statements are equivalent:*

- *The parallelotope is irreducible.*
- *The canonical scaling of the tiling is unique up to a common multiplier.*

8.3 Proof of theorem 13

(1) In the first statement of the theorem, we establish the structure of the Venkov graph of $P = P_1 \oplus P_2$. We have $\mathrm{Vert}(V(P)) = \mathrm{Vert}(V_1) \sqcup \mathrm{Vert}(V_2)$, where V_1, V_2 are subgraphs of $V(P)$ induced by vertices of the types $(\pm F_1 \oplus P_2)$, $(\pm F_2 \oplus P_1)$ where $F_1 \prec P_1$, $F_2 \prec P_2$ are any facets of P_1 and P_2.

We first prove that V_1 is isomorphic to $V(P_1)$, with isomorphism given by

$$(\pm F_1) \to (\pm F_1 \oplus P_2) \tag{8.1}$$

In other words, $(\pm F_1 \oplus P_2)$ and $(\pm F_2 \oplus P_2)$ are connected by a red/blue edge in $V(P)$ if and only if $(\pm F_1)$ and $(\pm F_2)$ are connected by a red/blue edge in $V(P_1)$.

A belt on parallelotope P_1, say, a hexagonal belt $\pm F_1, \pm G_1, \pm H_1$ produces the belt $(\pm F_1 \oplus P_2), (\pm G_1 \oplus P_2), (\pm H_1 \oplus P_2)$ on P. Therefore these three vertices of $V(P)$ are connected by a red triangle. Conversely, a hexagonal belt $(\pm F_1 \oplus P_2), (\pm G_1 \oplus P_2), (\pm H_1 \oplus P_2)$ on P is defined by a $(d-2)$-face $F^{d_1-2} \oplus P_2$, where $F^{d_1-2} \prec P_1$, $d_1 = \dim(P_1)$, therefore $\pm F_1, \pm G_1, \pm H_1$ is a belt on P_1, so $(\pm F_1), (\pm G_1), (\pm H_1)$ are connected by a red triangle in $V(P_1)$.

The case of quadruple belts is identical. We have proved that equation (8.1) indeed establishes an isomorphism of graphs V_1 and $V(P_1)$. Similar argument applies to V_2 and $V(P_2)$.

Next, consider vertices from different components: $(\pm F_1 \oplus P_2) \in \mathrm{Vert}(V_1)$ and $(\pm F_2 \oplus P_1) \in \mathrm{Vert}(V_2)$, where $F_1 \prec P_1$, $F_2 \prec P_2$ are facets. The two pairs of facets form a common quadruple belt, defined by the $(d-2)$-face $F_1 \oplus F_2$, therefore the vertices are joined by a blue edge in $V(P)$.

We have proved statement 1 of the theorem.

(2) We now prove the second statement. We will need lemma 23 and theorem 14 below, which connect local and global convexity of a polytope collection.

Definition 9 *A subset A in $\mathbb{R}^d$ is called locally convex at $x \in A$ if there is an open neighborhood U of x such that $A \cap U$ is convex. A subset A is called locally convex if it is locally convex at each point $x \in A$.*

Lemma 23 *([Bea83]) Let E be a locally convex closed subset of a plane, let points $u, v, w \in E$ be such that the segments $[u, v]$,$[v, w]$ are contained in E. Then the segment $[u, w]$ is also contained in E.*

Theorem 14 *Let $d \geq 3$, K be a locally finite collection of d-dimensional polytopes in $\mathbb{R}^d$, and let $S = \cup_{P \in K} P$. Then S is convex if and only if* $\mathrm{int}(S)$ *is connected and S is locally convex at points of* $\mathrm{relint}(F)$ *for all $(d-1)$, $(d-2)$-faces F of polytopes in K.*

Proof. Call two points $x, y \in \mathrm{int}(S)$ broken line connected if they can be connected by a broken line contained in $\mathrm{int}(S)$. This relation is an equivalence, and the equivalence classes are open. Since $\mathrm{int}(S)$ is connected, there is just one equivalence class. Also note that S is closed since K is locally finite.

Take two points $x, y \in S$. They belong to some polytopes from K. Since the polytopes are d-dimensional, for every $\epsilon > 0$ in ϵ-neighborhoods of x,y there are points x',y' of $\mathrm{int}(S)$.

Choose a broken line l contained in $\mathrm{int}(S)$ with vertices $x' = x_1, \ldots, x_n = y'$. Let $\Pi_k = \mathrm{aff}(x_1, x_k, x_{k+1})$, let L_k be the linear space so that $\Pi_k = L_k + x_1$. Consider the collection of affine spaces

$$\mathcal{C} = \{\mathrm{aff}(F^{d-3}) : F^{d-3} \prec P, P \in K\}.$$

The collection $\mathcal{C}$ is at most countable. We choose a vector $t \in \mathbb{R}^d$, $|t| < \epsilon$ so that the broken line $l + t$ is contained in $\mathrm{int}(S)$ and for all $k = 2, \ldots, n-1$ and $A \in \mathcal{C}$ the affine spaces $\Pi_k + t$ and A do not intersect:

$$A \cap L_k + x_1 + t = \emptyset$$

which is equivalent to

$$x_1 + t \notin A + L_k.$$

We can do that since the affine spaces $A + L_k$ are of dimension $d - 1$ or less, and there are at most countable number of them. By translating the broken line l by the vector t we assume that the affine spaces Π_k and A, $k = 2, \ldots, n-1$, $A \in \mathcal{C}$ do not intersect. Note that $|x_1 - x| < 2\epsilon$, $|x_n - y| < 2\epsilon$.

We proceed to prove that the segments $[x_1, x_k]$ are contained in S, arguing by induction. For $k = 2$ the claim is true since $[x_1, x_2]$ is a segment of the broken line. Suppose we have proved that $[x_1, x_k] \subset S$. If the points x_1, x_k, x_{k+1} are collinear, we are done. Otherwise consider the 2-dimensional affine space Π_k. Every point x of the intersection $S \cap \Pi_k$ is a relative interior point of a d, $(d-1)$ or $(d-2)$-face of some polytope $P \in K$, so by the hypothesis of the theorem S is locally convex at x, hence $S \cap \Pi_k$ is locally convex at x. Therefore $S \cap \Pi_k$ is a locally convex set. $S \cap \Pi_k$ is closed, and $[x_1, x_k], [x_k, x_{k+1}] \subset S \cap \Pi_k$. It follows from the lemma 23 that $[x_1, x_{k+1}] \subset S \cap \Pi_k \subset S$. This proves the induction step.

We have proved that the segment $[x_1, x_n]$ is contained in S. Since S is closed and an arbitrary ϵ was chosen, we have $[x, y] \subset S$. The theorem is proved. $\square$

We now continue to prove the second statement of theorem 13.

The *facet vector* of a parallelotope corresponding to facet F is the vector which shifts the parallelotope onto its neighbor across F.

Let $\mathcal{F}$, $\mathcal{N}$ be the sets of facet vectors and facet normals of the parallelotope. $\mathcal{F}$ is a centrally symmetric set of nonzero vectors. We have $\mathcal{F} = \mathcal{F}_1 \sqcup \mathcal{F}_2$ where $\mathcal{F}_1$, $\mathcal{F}_2$ are the sets of facet vectors corresponding to the two components of the Venkov graph. Let $\mathcal{N}_1$, $\mathcal{N}_2$ be the sets of corresponding facet normals.

Fix a normal tiling of $\mathbb{R}^d$ by translated copies of P such that one of the parallelotopes is centered at 0. Let Λ be the lattice of the tiling. For $x \in \Lambda$, $P(x)$ will stand for the parallelotope of the tiling with the center x.

(A) First we prove that the lattices $\mathbb{Z}(\mathcal{F}_1)$ and $\mathbb{Z}(\mathcal{F}_2)$ have no common points except 0. Take $\alpha_1, \alpha_2 > 0$, $\alpha_1 \neq \alpha_2$. To every ordered pair $[Q_1, Q_2]$ of adjacent parallelotopes of the tiling, we assign a vector in $\mathbb{R}^d$ as follows. Take the facet vector f so that $Q_2 = Q_1 + f$, and let $T[Q_1, Q_2] = \alpha_i f$ where i is defined by $f \in \mathcal{F}_i$. This assignment allows to define the gain function for an arbitrary combinatorial path $[Q_1, \dots, Q_n]$ in the tiling:

$$T[Q_1, \dots, Q_n] = T[Q_1, Q_2] + \dots + T[Q_{n-1}, Q_n]. \tag{8.2}$$

The gain function is equal to 0 for $(d-2)$-primitive circuits. It follows from the fact that the star of a hexagonal $(d-2)$ face has only three facets of one class, and the star of a quadruple $(d-2)$-face has 4 parallelotopes centered in the vertices of a parallelogram dual cell, where parallel edge vectors of the parallelogram belong to the same class.

By the quality translation theorem for the additive group $\mathbb{R}^d$ (a version of theorem 6 on page 24), the gain on all combinatorial circuits is equal to 0. Therefore we can assign a vector to each parallelotope as follows: assign 0 to parallelotope P. Then connect each parallelotope P' with P by a combinatorial path $[Q_1, \dots, Q_n]$, where $P = Q_1$, $P' = Q_n$, and assign vector $T[Q_1, \dots, Q_n]$ to P'.

If $\lambda \in \mathbb{Z}(\mathcal{F}_i)$, then the parallelotope $P(\lambda)$ is achievable by a combinatorial path which starts at P and only crosses facets of the class i. Hence the vector assigned to λ is $\alpha_i \lambda$. This proves that so long as $\lambda \neq 0$, λ cannot belong to both lattices.

(B) The linear spaces $\mathrm{lin}(\mathcal{F}_1)$ and $\mathrm{lin}(\mathcal{F}_2)$ are complementary:

$$\mathbb{R}^d = \mathrm{lin}(\mathcal{F}_1) \oplus \mathrm{lin}(\mathcal{F}_1).$$

To prove this, think of the lattice of the tiling as $\mathbb{Z}^d$. Let $d_i = \dim(\mathrm{lin}(\mathcal{F}_i))$. Suppose that $d_1 + d_2 > d$. Choose bases $f_1, \ldots, f_{d_1} \in \mathcal{F}_1$, $g_1, \ldots, g_{d_2} \in \mathcal{F}_2$ for the linear spaces $\mathrm{lin}(\mathcal{F}_1)$ and $\mathrm{lin}(\mathcal{F}_2)$. Note that the chosen vectors have integer components. Consider two linear subspaces $\mathbb{Q}(f_1, \ldots, f_{d_1})$, $\mathbb{Q}(g_1, \ldots, g_{d_2})$ of the linear space $\mathbb{Q}^d$ over $\mathbb{Q}$. Since $d_1 + d_2 > d$, the two subspaces have a nonzero common point $x \in \mathbb{Q}^d$. Point x can be represented in two ways:

$$x = r_1 f_1 + \cdots + r_{d_1} f_{d_1},$$

$$x = s_1 g_1 + \cdots + s_{d_2} g_{d_2}$$

where the numbers r_i and s_i are rational. Let D be a common multiple of the denominators of the numbers r_i, s_i, then the point Dx has integer coordinates in both bases. Hence it belongs to both lattices $\mathbb{Z}(\mathcal{F}_1)$ and $\mathbb{Z}(\mathcal{F}_2)$ which is impossible by the previous argument.

We have proved that $d_1 + d_2 \leq d$. Since $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ spans $\mathbb{R}^d$, we have $d_1 + d_2 \geq d$. Therefore $d_1 + d_2 = d$. The two linear spaces $\mathrm{lin}(\mathcal{F}_1)$, $\mathrm{lin}(\mathcal{F}_2)$ do not intersect, otherwise they would not together generate $\mathbb{R}^d$.

(C) Let K_1 be the complex of parallelotopes $P(x)$ where $x \in \mathbb{Z}(\mathcal{F}_1)$, let $S_1 = \cup_{P \in K_1} P$. We prove that S_1 is convex.

Since the lattice $\mathbb{Z}(\mathcal{F}_1)$ is generated by facet vectors, each two polytopes of K_1 can be connected by a combinatorial path consisting of polytopes from K_1. It follows that $\mathrm{int}(S)$ is connected.

Since the parallelotope tiling is normal, S is locally convex at points of relative interior of $(d-1)$-faces if K_1. Indeed, there are two types of $(d-2)$-faces.

Consider a hexagonal $(d-2)$-face F^{d-2} of K_1. There are 3 parallelotopes in its star. Either all three belong to K_1, or only one, say, $P+\lambda$. In the first case the relative internal points of F^{d-2} are internal points of S_1, in the second case every point $x \in \mathrm{relint}(F^{d-2})$ has a neighborhood $U = B_\epsilon(x)$ such that $U \cap S_1 = U \cap (P+\lambda)$, so x is a local convexity point of S_1.

Consider a quadruple $(d-2)$-face F^{d-2} of K_1. There are 4 parallelotopes in its star with respect to the tiling whose centers are λ, $\lambda+f$, $\lambda+g$, $\lambda+f+g$ where $\lambda \in \mathbb{Z}(\mathcal{F}_1)$, f, g are facet vectors. Let $\mathbf{m}$, $\mathbf{n}$ be the outward normal vectors to facets of $P+\lambda$ which correspond to facet vectors f, g, chosen so that

$$f \cdot \mathbf{m} = g \cdot \mathbf{n} = 2 \tag{8.3}$$

We have

$$f \cdot \mathbf{n} = g \cdot \mathbf{m} = 0 \tag{8.4}$$

Four cases are possible.

1. $f, g \in \mathcal{F}_1$. Then all of the four points λ, $\lambda+f$, $\lambda+g$, $\lambda+f+g$ belong to $\mathbb{Z}(\mathcal{F}_1)$, so $\mathrm{relint}(F^{d-2}) \subset \mathrm{int}(S_1)$.

2. $f, g \in \mathcal{F}_2$. Since $\mathbb{Z}(\mathcal{F}) = \mathbb{Z}(\mathcal{F}_1) \oplus \mathbb{Z}(\mathcal{F}_2)$, the points $\lambda+f$, $\lambda+g$, $\lambda+f+g$ do not belong to $\mathbb{Z}(\mathcal{F}_1)$. Every point $x \in \mathrm{relint}(F^{d-2})$ has a neighborhood $U = B_\epsilon(x)$ such that $U \cap S_1 = U \cap (P+\lambda)$, hence the set S_1 is locally convex at x.

3. $f \in \mathcal{F}_1$, $g \in \mathcal{F}_2$. We have $\lambda + f \in \mathbb{Z}(\mathcal{F}_1)$, $\lambda + g \notin \mathbb{Z}(\mathcal{F}_1)$, $\lambda + f + g \notin \mathbb{Z}(\mathcal{F}_1)$.

For every point $x \in \text{relint}(F^{d-2})$ there is a neighborhood $U = B_\epsilon(x)$ such that

$$\begin{aligned} S_1 \cap U &= (P(\lambda) \cup P(\lambda + f)) \cap U, \\ P(\lambda) \cap U &= \{x : (x - \lambda) \cdot \mathbf{n} \leq 1, (x - \lambda) \cdot \mathbf{m} \leq 1\} \cap U, \\ P(\lambda + f) \cap U &= \{x : (x - \lambda - f) \cdot \mathbf{n} \leq 1, (x - \lambda - f) \cdot (-\mathbf{m}) \leq 1\} \cap U. \end{aligned} \tag{8.5}$$

Note that sets $\{x : (x - \lambda) \cdot \mathbf{m} \leq 1\}$ and $\{(x - \lambda - f) \cdot (-\mathbf{m}) \leq 1\}$ are actually two half-spaces with the same bounding hyperplane, hence

$$S_1 \cap U = \{x : (x - \lambda) \cdot \mathbf{n} \leq 1\} \cap U. \tag{8.6}$$

It follows that the set S_1 is locally convex at x.

4. $f \in \mathcal{F}_2$, $g \in \mathcal{F}_1$. This case is identical to the previous one.

We have proved that the set S_1 is locally convex at relative interior points of $(d-1)$ and $(d-2)$-faces of the polytopes from the complex K_1. By theorem 14, set S_1 is convex.

Note that S_1 is invariant under the lattice $\mathbb{Z}(\mathcal{F}_1)$. It follows that it is invariant under translations from the linear space $\mathbb{R}(\mathcal{F}_1)$. Indeed, consider a facet F^{d-1} of the parallelotope P with the facet vector $f \in \mathcal{F}_2$. Let $\mathbf{n}$ be the corresponding facet normal. We prove that $\mathbf{n}$ is orthogonal to $\mathbb{R}(\mathcal{F}_1)$. Suppose there is a vector $a \in \mathbb{R}(\mathcal{F}_1)$ such that $a \cdot \mathbf{n} > 0$. Let $x \in \text{relint}(F^{d-1})$. Since $P + f$ does not belong to K_1, for small enough $t > 0$ we have $x + at \in \text{int}(P + f)$, so $x + at \notin S_1$. The contradiction proves that vector sets $\mathcal{N}_2$ and $\mathcal{F}_1$ are orthogonal. Symmetrically, $\mathcal{N}_1$ is orthogonal to $\mathcal{F}_2$. Hence $\mathcal{N} = \mathcal{N}_1 \sqcup \mathcal{N}_2$ is a decomposition into a direct sum.

There is the corresponding corresponding decomposition of the parallelotope P

into a direct sum of polytopes $P = P_1 \oplus P_2$. Polytope P_1 can be found by projecting P along $\mathrm{lin}(\mathcal{F}_2)$ onto $\mathrm{lin}(\mathcal{F}_1)$; similarly, P_2 is the projection of P along $\mathrm{lin}(\mathcal{F}_1)$ onto $\mathrm{lin}(\mathcal{F}_2)$.

By Venkov conditions, P_1 and P_2 are parallelotopes. Indeed, both P_1 and P_2 are centrally symmetric, and have centrally symmetric facets. Each belt of one of the polytopes can be lifted to P, therefore all belts on P_1 and P_2 are either hexagonal or quadruple, so polytopes P_1 and P_2 satisfy the Venkov conditions.

This completes the proof of the second statement.

Chapter 9

Conclusion

The value of many hard problems is often not in the problem itself, but in the theories and side results that come out of it. From Voronoi's conjecture on parallelotopes, originated the idea of quality translation, a number of results on zonotopes, and generally a better understanding of the geometry of polytopes.

We now look back at the results we have obtained, discuss ways to resolve the Voronoi conjecture, and pose some questions which we couldn't answer.

The idea of "coherent" parallelogram dual cells was very effective in proving the Voronoi conjecture for 3-irreducible tilings. The existence of incoherent parallelograms would disprove the Voronoi conjecture on parallelotopes. The most important result was that incoherent parallelograms in a dual 4-cell can only come in groups: a vertex of an incoherent parallelogram must belong to another incoherent parallelogram.

Our definition of an incoherent parallelogram was relative to a dual 4-cell. Robert Erdahl suggested another definition. Consider a parallelogram dual cell Π. Given any two facets F_1^{d-1}, F_2^{d-1} which correspond to the edges of the parallelogram, it follows from our theorem 2 that there are combinatorial paths on the $(d-1)$-skeleton of the tiling which connect F_1^{d-1}, F_2^{d-1} and where the $(d-2)$-joints are all hexagonal. This path can be used to transfer the scale factor from F_1^{d-1} to F_2^{d-1}. In the new definition, the parallelogram Π is called *coherent* if all such combinatorial paths lead to the same scale factor on F_2^{d-1}, and if the two facets are translates of each other, then the scale factor on F_2^{d-1} is the same as on F_1^{d-1}.

Existence of incoherent parallelograms in the new sense would contradict the Voronoi conjecture. Their nonexistence, however, will prove the Voronoi conjecture.

C0. *Can we transfer our results on incoherent parallelograms for the case of the new definition?*

Below is a list of other statements that we are very interested in, but couldn't prove.

C1. *The dimension of a dual cell and the dimension of the corresponding face of the tiling sum to d.* We have neither lower nor upper bound.

C2. *Dual cells form a polyhedral complex.* So far we could only prove that dual cells cover the space. The proof can be found in the thesis proposal document, which can be downloaded from the author's web site *http://www.mast.queensu.ca/ ˜ordine.*

C3. *A 'local' version of the Venkov graph criterion.* Let F^{d-k} be a face of the tiling. Facets in the star of F^{d-k} define a subgraph $V_{F^{d-k}}(P)$ of the Venkov graph to the parallelotope. The following statements are equivalent:

1. $V_{F^{d-k}}(P)$ is connected by red edges

2. The dual cell corresponding to F^{d-k} is irreducible (ie. it cannot be represented as a direct Minkowski sum of polytopes of smaller positive dimensions)

3. The tiling is locally reducible at F^{d-k}.

C4. *Equivalence of two versions of tiling irreducibility.* The parallelotope of a d-reducible tiling can be represented as a direct Minkowski sum of parallelotopes of smaller positive dimensions.

C5. *A regularity condition for dual cells.* Consider a dual 4-cell. We do not know if the dimension of a dual 4-cell (as a polytope) is actually 4. However, we know the full classification of dual 3-cells. Therefore the boundary of a dual 4-cell can be considered as an immersion of a 3-dimensional polyhedral complex in the d-dimensional space. What are properties of the immersion? Is it an embedding?

We think that a proof (or a counterexample) to the Voronoi conjecture can be obtained by considering increasingly large classes of tilings: 2-irreducible, 3-irreducible, 4-irreducible and so on, and trying to devise an induction scheme. We conjecture that a d-irreducible tiling of the d-dimensional space has a reducible parallelotope $P = P_1 \oplus P_2$. Then it would be sufficient to solve the conjecture for parallelotopes P_1, P_2 of smaller dimensions.

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