Finitely presented semigroups and groups in knot theory

I. A. Dynnikov*

Moscow State University
Vorobyovy gory, Moscow 119899
e-mail: dynnikov@mech.math.msu.su

Abstract

In this paper we construct finitely presented semigroups whose central elements are in one-to-one correspondence with isotopy classes of non-oriented links in $\mathbb{R}^3$. Solving the word problem for those semigroups is equivalent to solving the classification problem for links and tangles. Also, we give a construction of finitely presented groups containing the braid group as a subgroup.

1. Introduction

By a link in $\mathbb{R}^3$ we shall mean the union of a number of pairwise non-intersecting simple closed polygonal arcs in $\mathbb{R}^3$. In this paper we shall consider only *non-oriented* links. Two links are called equivalent if they are ambiently isotopic in $\mathbb{R}^3$. Let us denote by L_n, $n = 0, 1, 2, \ldots$, the set of isotopy classes of n-component links. The set L_0 consists of exactly one element, the empty link; L_1 is the set of isotopy classes of all knots in $\mathbb{R}^3$.

Let

$$L = L_0 \cup L_1 \cup L_2 \cup \ldots$$

be the set of isotopy classes of all links. L carries natural structure of an Abelian semigroup with multiplication given by taking the disjoint union of two links separated by a plane and with the empty link as the unit. The semigroup L is not finitely generated. As an Abelian semigroup it is freely generated by the isotopy classes of so-called nonsplit links.

In the present paper we construct a series of semigroups Y_n, $n = 3, 4, \ldots, \infty$, possesing the following properties:

1) the center $Z(Y_n)$ of the semigroup Y_n is isomorphic to the semigroup L;
2) the group D_n of invertible elements of Y_n contains a subgroup isomorphic to the braid group of infinitely many strands;
3) for $n < \infty$ the semigroup Y_n and the group D_n are finitely presented ones, i.e. they can be presented by a finite number of generators and defining relations.

Constructing a semigroup having only properties 1) and 2) is easy. We shall do this in Section 2 by modifying slightly the standard construction of the category of non-oriented tangles. A decomposition of an element of that semigroup into a product of the generators is given by any generic diagram of the corresponding tangle, the defining relations express isotopies of diagrams and the Reidemeister moves. Those relations, more preceisely their analogues for the category of oriented tangles, have been used in [1] for constructing link invariants.

The main difference of the so constructed semigroup T from the semigroups Y_n, $n = 3, 4, \ldots$, is in that it can not be presented by finitely many defining relations. However, the semigroup T can be embedded in each of the semigroups Y_n, which are finitely presented.

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In this way we reduce the topological problem of recognizing links to a purely algebraic one, the word problem for a finitely presented semigroup. According to paper by S. V. Matveev [2], an algorithm for recognizing links exists. Moreover, as S. V. Matveev told the author, by enhancing that algorithm , one can construct an algorithm for recognizing tangles, and hence, one can solve the word problem for the semigroups Y_n. However, it would be interesting to see a purely algebraic proof of solvability of the word problem for those semigroups and to obtain in this way an alternative proof of recognizibility of links.

2. Tangles

Recall the construction of the category of (non-oriented) tangles [1]. A tangle of type (k, l) is an arbitrary one-dimensional compact PL-submanifold t in $\mathbb{R}^3$ lying in the region

$$\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 \,|\, 0 \le x \le 1\} \tag{1}$$

and having the set $\{(0, i, 0)\}_{1 \le i \le k} \cup \{(1, j, 0)\}_{1 \le j \le l} = t \cap \partial\mathcal{S}$ as its boundary. Two tangles are called equivalent if they are ambiently isotopic in $\mathcal{S}$ assuming that all the boundary points of $\mathcal{S}$ are fixed under isotopy. Objects of the tangle category $\mathcal{T}$ are non-negative integers, morphisms from the object l to the object k are isotopy classes of tangles of type (k, l).

The composition $t_1 \circ t_2$ of two arbitrary tangles t_1 and t_2 of type (k, l) and (l, m) respectively is defined as the union

$$\psi_1(t_1) \cup \psi_2(t_2),$$

where $\psi_{1,2}$ are the maps from $\mathcal{S}$ to itself defined by

$$\psi_1(x, y, z) = (x/2, y, z), \quad \psi_2(x, y, z) = ((x + 1)/2, y, z).$$

In this way, the composition of morphisms of the category $\mathcal{T}$ is well defined. From the point of view of this category the semigroup L is the semigroup of morphisms of the object 0 to itself.

By modifying definitions slightly, we can obtain a semigroup, a simpler object than the tangle category. To this end, let us change the definition of tangle in the following way. We shall call a tangle any PL-submanifold t in $\mathcal{S}$ whose boundary ∂t is the set

$$\{(0, i, 0), (1, i, 0)\}_{i>0}$$

with $t \cap \partial\mathcal{S} = \partial t$, and such that all its components lying far enough from the origin are straight line segments of the form $[(0, i, 0), (0, j, 0)]$ (we shall denote the straight line segment connecting points A and B by $[A, B]$). Under this condition the difference $j - i$ is constant for large i. Objects of this kind could be called tangles of type (∞, ∞). An example of a tangle in the sence of the just given definition is shown in fig. 1.

Notice that formally speaking, the composition of two tangles may now not be a tangle, but it is a PL-submanifold ambiently isotopic to a tanlge in $\mathcal{S}$. Hence the composition is well defined on the set T of isotopy classes of such tangles. Endowed with this operation the set T becomes a semigroup.

There is an evident functor F from the category $\mathcal{T}$ to the semigroup T. It is defined as follows. Let f be a morphism from an object l to k. The morphism f can be presented by a tangle (in the standard sence) lying by whole in the region

$$y \le k + x(l - k).$$

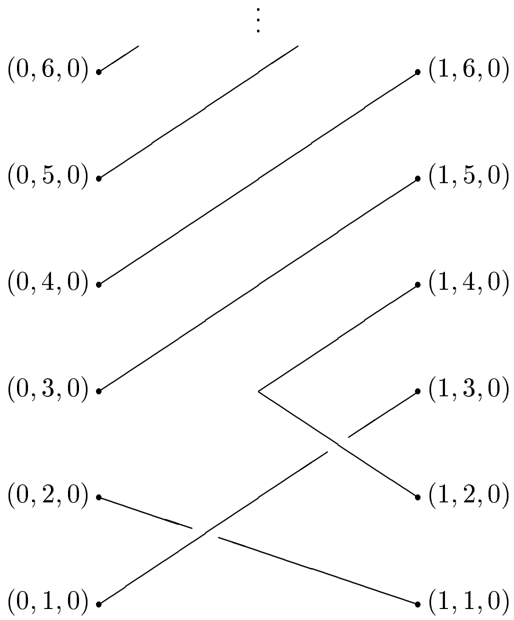
Let us add to this tangle the segments $[(0, k + i, 0), (1, l + i, 0)]$, $i > 0$. The isotopy class of the obtained tangle (in the new sence) is what we count as the image $F(f)$ of the morphism f.

The proof of the following assertion is easy.

Lemma 1 *Tangles t_1 and t_2 of type (k, l) are equivalent if and only if their images $F(t_1)$ and $F(t_2)$ coincide. An element of the semigroup T has the form $F(t)$, where t is a link (i.e. a tangle of type $(0, 0)$), if and only if it is central.*

It is easy to present the semigroup T by generators and defining relations.

Figure 1: A tangle of type (∞, ∞)



Lemma 2 *The semigroup T is generated by the elements $\alpha_i, \beta_i, \sigma_i, \sigma_i^{-1}$, $i > 0$, shown in fig. 2. The following relations holds in T:*

$$\alpha_i \beta_{i+1} = 1, \tag{2}$$
$$\alpha_{i+1} \beta_i = 1, \tag{3}$$
$$\alpha_i \alpha_{i+j+1} = \alpha_{i+j-1} \alpha_i \tag{4}$$
$$\alpha_i \beta_{i+j+1} = \beta_{i+j-1} \alpha_i, \tag{5}$$
$$\beta_i \beta_{i+j-1} = \beta_{i+j+1} \beta_i \tag{6}$$
$$\beta_i \alpha_{i+j-1} = \alpha_{i+j+1} \beta_i, \tag{7}$$
$$\alpha_i \sigma_{i+j+1} = \sigma_{i+j-1} \alpha_i, \tag{8}$$
$$\alpha_{i+j+1} \sigma_i = \sigma_i \alpha_{i+j+1}, \tag{9}$$
$$\beta_i \sigma_{i+j-1} = \sigma_{i+j+1} \beta_i, \tag{10}$$
$$\beta_{i+j+1} \sigma_i = \sigma_i \beta_{i+j+1}, \tag{11}$$
$$\sigma_i \sigma_{i+j+1} = \sigma_{i+j+1} \sigma_i, \tag{12}$$
$$\alpha_i \sigma_{i+1}^{\pm 1} \beta_{i+2} = \sigma_i^{\mp 1}, \tag{13}$$
$$\alpha_{i+2} \sigma_{i+1}^{\pm 1} \beta_i = \sigma_i^{\mp 1}, \tag{14}$$
$$\alpha_i \sigma_{i+1}^{\pm 1} \beta_i = 1, \tag{15}$$
$$\alpha_{i+1} \sigma_i^{\pm 1} \beta_{i+1} = 1, \tag{16}$$
$$\sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = 1, \tag{17}$$
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \tag{18}$$

where $i, j > 0$. These relations can be taken as a set of defining relations of the semigroup T.

This assertion claims nothing new compared with a similar statement about defining relations of the tangle category (see [1]). Relations (2)–(14) correspond to isotopies of tangle

Figure 2: Generators of the semigroup T

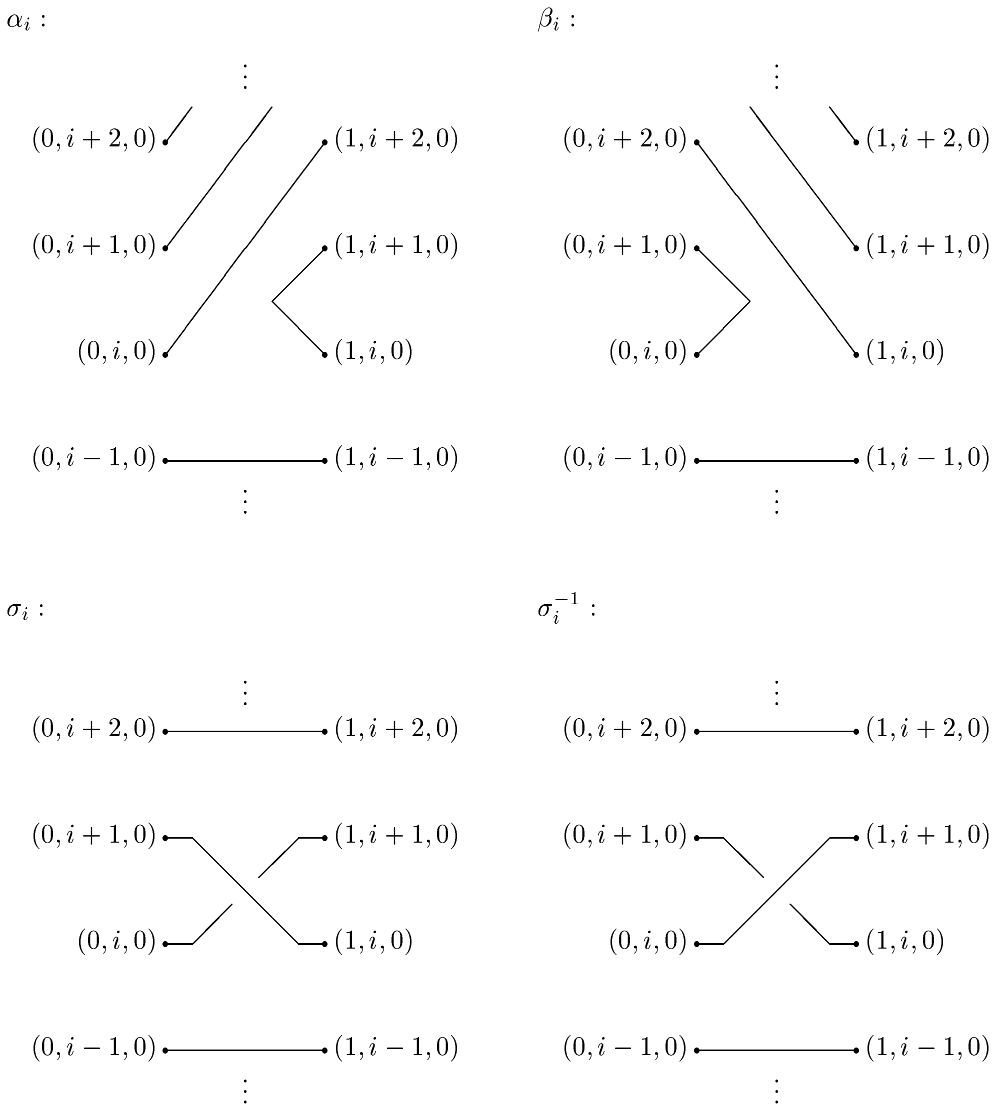
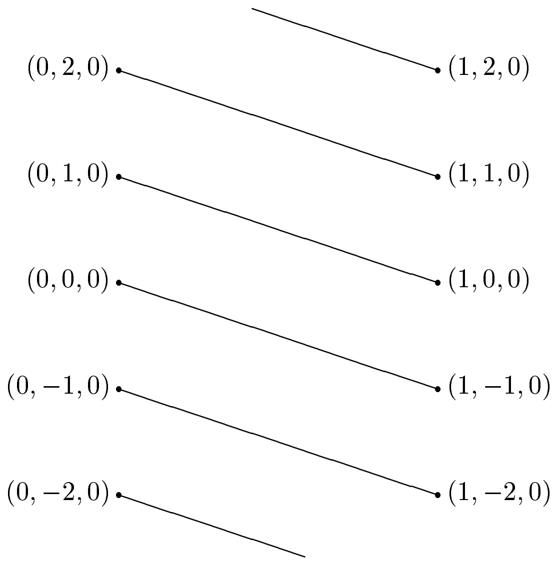


Figure 3: The tangle τ



diagrams, relations (15), (16) to the Reidemeister move of type I, relations (17) and (18) to the Reidemeister moves of type II and III respectively.

Remark. The notion of tangle could be generalized a little differently, by adding the points $(0, i, 0), (1, i, 0)$ with $i \leq 0$ to the boundary. In this case, the obtained semigroup of tangles is finitely generated due to the relations

$$\begin{aligned} \tau^i \alpha_1 \tau^{-i} &= \alpha_{i+1}, \\ \tau^i \beta_1 \tau^{-i} &= \beta_{i+1}, \\ \tau^i \sigma_1^{\pm 1} \tau^{-i} &= \sigma_{i+1}^{\pm 1}, \end{aligned}$$

where τ is the tangle shown in fig. 3. But there does not exist a finite presentation of this semigroup.

3. Tangles embedded in book

Here we shall define a semigroup Y_∞, into which the semigroup T can be embedded. The semigroup Y_∞ itself is not finitely presented, but we will need it for the further construction of finitely presented semigroups which are subsemigroups of Y_∞.

Denote by P_k $(k \in \mathbb{Z})$ the half-plane

$$\{(x, y, z) \in \mathbb{R}^3 \,|\, y \geq 0, z = ky\} \tag{19}$$

and consider the part of the union of these half-planes that lies in the region (1):

$$\mathbb{B} = \mathcal{S} \cap \Big(\bigcup_{k \in \mathbb{Z}} P_k \Big). \tag{20}$$

It is natural to call the set $\mathbb{B}$ a book and the line $y = z = 0$ the binding line. Let us denote by $E^0_{k,l}$ and $E^1_{k,l}$, where $k, l \in \mathbb{Z}$, $l > 0$, the following points lying at the boundary of the book $\mathbb{B}$:

$$E^0_{k,l} = (0, l, kl), \quad E^1_{k,l} = (1, l, kl). \tag{21}$$

By a tangle in the book $\mathbb{B}$ we shall call any PL-submanifold t in $\mathcal{S}$, verifying the following conditions:

1) $\partial t = t \cap \partial \mathcal{S} = \{E^0_{k,l}, E^1_{k,l}\}_{k,l\in\mathbb{Z},l>0}$;

2) t is contained entirely in $\mathbb{B}$;

3) t is transversal to the binding line, i.e. it has only a finite number of intersecion points with l which do not disappear under small variation of the tangle t without violating condition 2;

4) the restriction of the x-coordinate to any connected component of the intersection $t \cap P_i$ is a monotonic function for any $i \in \mathbb{Z}$.

We shall call two tangles of this kind equivalent if they are ambiently isotopic in $\mathcal{S}$ (but not necessarily in $\mathbb{B}$!).

The composition of tangles in the book $\mathbb{B}$ is defined in the same manner as in the case of ordinary tangles. The set of isotopy classes of tangles in $\mathbb{B}$ endowed with composition is a semigroup with the unit

$$I = \bigcup_{k,l\in\mathbb{Z},l>0} [E^0_{k,l}, E^1_{k,l}].$$

We shall denote this semigroup by Y_∞. Tangles in the book $\mathbb{B}$ will be called simply tangles in sequel.

For any $n \geq 3$ we denote by Y_n the subsemigroup in Y_∞ consisting of all isotopy classes of tangles t such that $t \cap \mathring{P}_k = I \cap \mathring{P}_k$ for $k < 0$ and for $k > n-1$. Here $\mathring{P}_k$ denotes the half-plane P_k with its boundary removed: $\mathring{P}_k = \{(x,y,z) \in \mathbb{R}^3 \,|\, y > 0, z = ky\}$.

Notice that any tangle t is isotopic to a tangle t' for which there exist $N_1, N_2 \in \mathbb{Z}$ such that $t' \cap \mathring{P}_k = I \cap \mathring{P}_k$ for $k < N_1$ and for $k > N_2$. In addition to this, the tangle t' can be chosen so that for sufficiently large N_3 and some integers n_k, $k \in \mathbb{Z}$, the segments $[E^0_{k,n}, E^1_{k,n+n_k}]$ are contained by whole in t' for $n > N_3$ and any $k \in \mathbb{Z}$. It is readily seen that the map $\varphi : Y_\infty \to \mathbb{Z}^{\mathbb{Z}}$

$$\varphi(t) = (\dots, n_{-2}, n_{-1}, n_0, n_1, n_2, \dots) \tag{22}$$

defined in this way is a group homomorphism.

It is natural to define the complexity $m(t)$ of a tangle t in the book $\mathbb{B}$ as the number of intersection points of t with the binding line. Let us consider the simplest nontrivial tangles, i.e. ones having complexity 1. If $m(t) = 1$, then it is easy to see that for exactly two values $i \in \mathbb{Z}$ the intersection $t \cap \mathring{P}_i$ is not isotopic to the intersection $I \cap \mathring{P}_i$. For those two exeptional values of i we have $\varphi(t)_i = \pm 1$. Moreover, if $\varphi(t)_i = 1$, then in the half-plane P_i the tangle t has the following connected componenets (up to isotopy):

$$t \cap P_i = \Big(\bigcup_{l>0} [E^0_{i,l}, E^1_{i,l+1}]\Big) \cup [A, E^1_{i,1}],$$

where A is the only intersection point of the tangle t with the binding line. In the case $\varphi(t)_i = -1$, $m(t) = 1$, we have (up to isotopy in P_i):

$$t \cap P_i = \Big(\bigcup_{l>0} [E^0_{i,l+1}, E^1_{i,l}]\Big) \cup [E^0_{i,1}, A].$$

Denote by a_i, b_i, c_i, d_i the elements of Y_∞ that are presented by tangles of complexity 1 such that the following holds:

$$\begin{aligned} \varphi(a_i)_j &= \delta_{i-1,j} + \delta_{i,j}, \\ \varphi(b_i)_j &= \delta_{i-1,j} - \delta_{i,j}, \\ \varphi(c_i)_j &= -\delta_{i-1,j} - \delta_{i,j}, \\ \varphi(d_i)_j &= -\delta_{i-1,j} + \delta_{i,j}. \end{aligned} \tag{23}$$

Here $\delta_{i,j}$ is the Kroneker delta.

Having drawn corresponding pictures, it is easy to check that the following relations take place int the semigroup Y_∞:

$$a_i d_{i+1} = a_{i+1} b_i, \tag{24}$$

$$d_i c_{i+1} = b_{i+1} c_i, \tag{25}$$

$$a_i d_i = a_{i+1} b_{i+1}, \tag{26}$$

$$b_i c_i = d_{i+1} c_{i+1}, \tag{27}$$

$$a_i b_i c_i d_i = 1, \tag{28}$$

$$b_i d_i = d_i b_i = 1, \tag{29}$$

$$x_i y_j = y_j x_i, \tag{30}$$

$$x_i d_{i-1} d_i d_{i+1} = d_{i-1} d_i d_{i+1} x_i \tag{31}$$

for all $i, j \in \mathbb{Z}$ such that $|i - j| > 1$ and $x_i \in \{a_i, b_i, c_i, d_i\}$, $y_j \in \{a_j, b_j, c_j, d_j\}$.

Theorem 1 *The elements $\{a_i, b_i, c_i, d_i\}_{i\in\mathbb{Z}}$ generate the semigroup Y_∞. Relations* (24)–(31) *can be taken as a set of defining relations of the semigroup Y_∞.*

Proof. To begin with, we prove the simpler part of the theorem, namely, the assertion that the semigroup Y_∞ is generated by the mentioned elements. Indeed, any tangle can be cut into tangles of complexity 1 by planes of the form $x = \text{const}$. Therefore, it is enough to show that any tangle of complexity 1 can be presented up to isotopy by a composition of our generators. Assume that for the tangle t of complexity $m(t) = 1$ we have

$$\varphi(t)_i = \varphi(t)_j = 1,$$

where $j > i$. Then it is isotopic to

$$a_{i+1} d_{i+2} d_{i+3} \cdots d_j.$$

If $m(t) = 1$ and

$$\varphi(t)_i = -\varphi(t)_j = 1,$$

$j > i$, then t is equivalent to the tangle

$$b_j b_{j-1} \cdots b_{i+1}.$$

The other two cases are similar.

The rest of this section is devoted to the proof of sufficientness of relations (24)–(31) for realizing any isotopy between two tangles in the book $\mathbb{B}$. A sketch of the proof look like this. According to relation (29) the elements b_i and d_i are each other inverse. Therefore, equivalence of two words w_1 and w_2 in the alphabet $A = \{a_i, b_i, c_i, d_i\}_{i\in\mathbb{Z}}$ modulo relations (24)–(31) takes place whenever so does equivalence of words $b_i w_1$ and $b_i w_2$ as well as $d_i w_1$ and $d_i w_2$, $w_1 b_i$ and $w_2 b_i$, $w_1 d_i$ and $w_2 d_i$. We shall show that for any word w one can find such words u and v containing only letters b_i and d_i with $i \in \mathbb{Z}$ that the word uwv is equivalent modulo relations (24)–(31) to a word presenting an element of a special subsemigroup in Y_∞ which is isomorphic to T. We shall prove that defining relations (2)–(18) of the semigroup T can be derived from (24)–(31).

For $i \in \mathbb{Z}$ we call a word $w = x_1 x_2 \ldots x_n$, where $x_1, \ldots, x_n \in A$, *i-balanced* if we have

$$\varphi(w)_i = 0$$

and for all $k = 1, \ldots, n$ the following enequality holds:

$$\varphi(x_1 \ldots x_k)_i \geq 0.$$

In the geometrical language, i-balanceness of the word w corresponding to a tangle $t \subset \mathbb{B}$ means that any connected component of the intersection $t \cap \overset{\circ}{P}_i$ connects either points of the form $E^0_{i,m}$ and $E^1_{i,m}$ or two vertices of the tangle t that lie at the binding line. For any $\varepsilon > 0$ each i-balanced tangle is ambiently isotopic in $\mathbb{B}$ to a tangle t' such that t' does not differ from the identity tangle in the half-plane $P_i \cap \{y \geq \varepsilon\}$.

The following interpretation of of the notion of i-balanced word is also convinient. For a given word w in the alphabet A and an integer i let us replace each letter x with a left parenthesis if $\varphi(x)_i = 1$, with a right parenthesis if $\varphi(x)_i = -1$, and with a space if $\varphi(x)_i = 0$. The word w is i-balanced if and only if the obtained parenthesis string is balanced.

Let $\widetilde{T}$ be the subsemigroup of Y_∞ consisting of all the isotopy classes of tangles that are i-balanced for all $i \neq 0$.

Lemma 3 *The semigroup $\widetilde{T}$ is isomorphic to the semigroup T that was introduced in Section 2.*

PROOF. It is easy to see that if a tangle t is i-balanced for all $i \neq 0$, then it can be transformed by an ambient isotopy in $\mathbb{B}$ to the form such that

$$t \cap \{|z| \geq 1/2\} = I \cap \{|z| \geq 1/2\}.$$

Forgetting the "trivial" part of the tangle t, which lies in the region $|z| \geq 1/2$, we obtain a tangle with boundary $\{E^0_{0,k}, E^1_{0,k}\}_{k>0}$, which defines an element from T. So, we get a map $\xi : \widetilde{T} \to T$. Clearly, this map is injective. We shall construct the inverse map. To this end we provide the images of the elements $\alpha_i, \beta_i, \sigma_i^{\pm 1}$, $i > 0$, generating the semigroup T, under the map ξ^{-1}:

$$\alpha_i \mapsto d_1^{i-1} a_1 b_1^i \tag{32}$$
$$\beta_i \mapsto d_1^i c_1 b_1^{i-1} \tag{33}$$
$$\sigma_i \mapsto d_1^{i-1} b_0 d_1 d_0 b_1^i \tag{34}$$
$$\sigma_i^{-1} \mapsto d_1^i b_0 b_1 d_0 b_1^{i-1}. \tag{35}$$

The fact that the tangles in the right hand side of these formulas are send under the map ξ into the generators $\alpha_i, \beta_i, \sigma_i^{\pm 1}$ shown in fig. 2 can be checked straightforwardly. In what follows we identify the semigroups $\widetilde{T}$ and T , considering T as a subsemigroup in Y_∞.

Lemma 4 *If a word w in the alphabet A is i-balanced for all $i \neq 0$, then it is equivalent up to relations* (24)–(31) *to a word of the form*

$$x_1 \dots x_m, \tag{36}$$

where $x_1, \dots, x_m \in \{d_1^{i-1} a_1 b_1^i, d_1^i c_1 b_1^{i-1}, d_1^{i-1} b_0 d_1 d_0 b_1^i, d_1^i b_0 b_1 d_0 b_1^{i-1}\}_{i>0}$.

PROOF. To begin with, we check that, applying relations (24)–(31) to the word w, we can get rid of all entries of letters a_i, b_i, c_i, d_i with $i \neq 0, 1$ in w. Denote by A_i the set $\{a_i, b_i, c_i, d_i\}$. Let N be the maximal i such that the word w contains letters from A_i, let $N > 1$. Relations (26), (27) and (29) implies:

$$a_N = a_N b_N d_N = a_{N-1} d_{N-1} d_N$$
$$c_N = b_N d_N c_N = b_N b_{N-1} c_{N-1}.$$

So, by applying those relations one can get rid of letters a_N, c_N in the word w. In the process, the word w will remain i-blanced for $i \neq 0$. Indeed, on the "bracket" language the mentioned replacements look like follows:

$$\begin{array}{lccccc} i = N-2: & \varnothing & \to & \varnothing & \to & (\,) \\ i = N-1: & (& \to & (\,(\,) & \to & (\,(\,) \\ i = N: & (& \to & (\,)\,(& \to & (\end{array},$$

$$\begin{array}{lccccc} i = N-2: & \varnothing & \to & \varnothing & \to & (\,) \\ i = N-1: &) & \to & (\,)\,) & \to & (\,)\,) \\ i = N: &) & \to &)\,(\,) & \to &) \end{array}.$$

Now we shall get rid of entries of letters b_N, d_N. Due to N-balanceness of the word w, its last letter cannot be d_N , and the number of b_N and d_N entries coincide. Let us find the last letter d_N in w. Let x be the letter following after d_N. There are the following three possibilities.

1) $x = b_N$. In this case we remove the subword $d_N b_N$.

2) $x \in A_{N-1}$. Make the following replacements:

$$d_N x \mapsto b_{N-1} b_{N-2} d_{N-2} d_{N-1} d_N x \mapsto b_{N-1} b_{N-2} x d_{N-2} d_{N-1} d_N.$$

3) $x \in A_i$, where $i \leq N-2$. Interchange d_N and x.

In all the three cases i-balanceness of the word w for $i \neq 0$ is preserved. In case 1) the number of d_N (as well as b_N) entries in w is decreased by 1. In cases 2) and 3) the number of d_N and w entries is preserved but the number of letters following after the last d_N is decreased. Hence after a finite number of such replacements we get a word not envolving letters b_N, d_N.

So, applying the described procedure we can change the word w so that it contains only letters from A_i, where $i \leq 1$. Similarly, using transforms

$$\begin{aligned}
a_N &\mapsto a_N d_N b_N \mapsto a_{N+1} b_{N+1} b_N, \\
c_N &\mapsto c_N b_N d_N \mapsto c_{N+1} d_{N+1} d_N, \\
b_N d_N &\mapsto \varnothing, \\
x d_N &\mapsto x d_N d_{N+1} d_{N+2} b_{N+2} b_{N+1} \mapsto d_N d_{N+1} d_{N+2} x b_{N+2} b_{N+1}, \quad x \in A_{N+1}, \\
x d_N &\mapsto d_N x, \quad x \in A_i, \quad i \geq N+2,
\end{aligned}$$

where $N < 0$, one can make the word w does not contain letters from A_i with $i < 0$. During this procedure we insert only only letters b_0, d_0, b_1, d_1 in the word w. By means of replacements

$$\begin{aligned}
a_0 &\mapsto a_0 d_0 b_0 \mapsto a_1 b_1 b_0, \\
c_0 &\mapsto c_0 b_0 d_0 \mapsto c_1 d_1 d_0
\end{aligned}$$

we obtain a word envolving only letters $b_0, d_0, a_1, b_1, c_1, d_1$ and still being i-balanced for all $i \neq 0$.

Thus, it suffices to prove the lemma in the case when the word w contains only letters $b_0, d_0, a_1, b_1, c_1, d_1$ and is (-1)-balanced and 1-balanced. By definition, (-1)-balanceness means that one obtain a well balanced parenthesis string by replacing all b_0's with a left bracket and all d_0's with a right bracket in the word w. We shall call the *depth* of the word w the maximal value of $\varphi(u)_{-1}$ taken over all possible decompositions of the word w into the product $w = uv$ of two subwords. In other words, the depth of w is the depth of the bracket hierarchy obtained by replacing each b_0 with a left bracket and each d_0 with a right one.

Our next goal is to make our word has depth 1 or 0 by applying relations (24)–(31). Suppose that the depth of the word w is greater than 1. Then for all $x, y \in A_1$ we shall implement the following replacements in w as long as possible:

$$\begin{aligned}
b_0 x y &\mapsto b_0 x d_0 b_0 y, \\
b_0^2 x d_0 b_0 &\mapsto b_0^2 x d_0^2 b_0^2, \\
b_0^2 x d_0 y &\mapsto b_0^2 x d_0^2 b_0 y.
\end{aligned}$$

Under these replacements, both (-1)-balanceness and the depth of w are preserved. It easy to see that since the depth of w is greater than 1 by assumption, after a finite number of such replacements, subwords of the form

$$b_0^2 x d_0^2, \quad \text{where } x \in A_1, \tag{37}$$

will necessarily appear. With each of those subwords we perform the following transforms, which are obtained by applying one ore more relations (24)–(31):

$$\begin{aligned}
b_0^2 a_1 d_0^2 &\mapsto b_0\big(b_0 a_1 (d_0 d_1 d_2) b_2 b_1\big) d_0\\
&\mapsto b_0\big(b_0 (d_0 d_1 d_2) a_1 b_2 b_1\big) d_0\\
&\mapsto b_0\big(d_1 d_2 a_1 b_2 b_1\big) d_0\\
&\mapsto b_0 b_{-1} d_{-1}\big(d_1 d_2 a_1 b_2 b_1\big) d_0\\
&\mapsto b_0 b_{-1}\big(d_1 d_2 a_1 b_2 b_1\big) d_{-1} d_0\\
&\mapsto b_0 b_{-1}\big(b_0 a_1 d_0\big) d_{-1} d_0\\
&\mapsto b_0 b_{-1} b_{-2} d_{-2}\big(b_0 a_1 d_0\big) d_{-1} d_0\\
&\mapsto b_0 b_{-1} b_{-2} b_0 a_1 d_0 d_{-2} d_{-1} d_0\\
&\mapsto b_0 b_{-1} b_{-2} b_0 a_0 d_0 d_1 d_0 d_{-2} d_{-1} d_0\\
&\mapsto b_0 b_{-1} b_{-2} b_0 a_0 b_{-1} (d_{-1} d_0 d_1) d_0 d_{-2} d_{-1} d_0\\
&\mapsto b_0 b_{-1} b_{-2} b_0 a_0 b_{-1} d_0 (d_{-1} d_0 d_1)(d_{-2} d_{-1}) d_0\\
&\mapsto b_0 b_{-1} b_{-2} b_0 a_0 b_{-1} d_0 d_{-1} d_0 (d_{-2} d_{-1}) d_1 d_0\\
&\mapsto b_0 b_{-1} b_{-2} b_0 a_0 b_{-1} d_0 d_{-1} d_0 (d_{-2} d_{-1} d_0 d_1) b_1 b_0 d_1 d_0\\
&\mapsto b_0 b_{-1} b_{-2} (d_{-2} d_{-1} d_0 d_1) b_0 a_0 b_{-1} d_0 d_{-1} d_0 b_1 b_0 d_1 d_0\\
&\mapsto d_1 b_0 a_0 b_{-1} d_0 d_{-1} d_0 b_1 b_0 d_1 d_0\\
&\mapsto d_1 b_0 a_0 b_{-1} d_0 (d_{-1} d_0 d_1) b_1 b_1 b_0 d_1 d_0\\
&\mapsto d_1 b_0 a_0 b_{-1} (d_{-1} d_0 d_1) d_0 b_1 b_1 b_0 d_1 d_0\\
&\mapsto d_1 b_0 (a_0 d_0 d_1) d_0 b_1 b_1 b_0 d_1 d_0\\
&\mapsto d_1 b_0 a_1 d_0 b_1^2 b_0 d_1 d_0;
\end{aligned}$$

by replacing in these formulas a_1 with d_1 and a_0 with b_0, we obtain a chain of tranforms that yields

$$b_0^2 d_1 d_0^2 \mapsto d_1 b_0 d_1 d_0 b_1^2 b_0 d_1 d_0;$$

by reversing the letter order in all the words and replacing a_i, b_i, c_i, d_i with c_i, d_i, a_i, b_i respectively, we get tranforms

$$b_0^2 c_1 d_0^2 \mapsto b_0 b_1 d_0 d_1^2 b_0 c_1 d_0 b_1,$$

$$b_0^2 b_1 d_0^2 \mapsto b_0 b_1 d_0 d_1^2 b_0 b_1 d_0 b_1,$$

which also are obtained by applying relations (24)–(31). All the mentioned tranforms preserve i-balanceness for $i \neq 0$. Having applied those transforms to all the subwords of the form (37), we get a word of a smaller depth. Repeating the procedure sufficiently many times we get a word of depth 1, i.e. a word decomposable into the product of subwords from the following lists:

$$a_1, \quad b_1, \quad c_1, \quad d_1, \quad b_0 a_1 d_0, \quad b_0 b_1 d_0, \quad b_0 c_1 d_0, \quad b_0 d_1 d_0.$$

By using tranforms

$$
\begin{aligned}
b_0a_1d_0 &\mapsto b_0(a_1d_2)b_2d_0\\
&\mapsto b_0(a_2b_1)b_2d_0\\
&\mapsto a_2b_0b_1b_2d_0\\
&\mapsto (a_2b_1)d_1b_0b_1b_2d_0\\
&\mapsto (a_1d_2)d_1b_0b_1b_2d_0\\
&\mapsto (a_1b_1)b_0(d_0d_1d_2)d_1b_0b_1b_2d_0\\
&\mapsto (a_0d_0)b_0d_1(d_0d_1d_2)b_0b_1b_2d_0\\
&\mapsto (a_0d_1)d_0d_1b_0d_2b_1b_2d_0\\
&\mapsto (a_1b_0)d_0d_1b_0b_1b_0(d_0d_1d_2)b_1b_2d_0\\
&\mapsto a_1d_1b_0b_1b_0b_1(d_0d_1d_2)b_2d_0\\
&\mapsto a_1d_1b_0b_1b_0b_1d_0d_1d_0\\
&\mapsto a_1d_1b_0b_1d_0(b_0^2b_1d_0^2)b_0d_1d_0\\
&\mapsto a_1d_1b_0b_1d_0(b_0b_1d_0d_1^2b_0b_1d_0b_1)b_0d_1d_0\\
&= a_1d_1(b_0b_1d_0)(b_0b_1d_0)d_1^2(b_0b_1d_0)b_1(b_0d_1d_0)
\end{aligned}
$$

and a similar chain resulting in the transform

$$b_0^2c_1d_0^2 \mapsto (b_0b_1d_0)d_1(b_0d_1b_0)b_1^2(b_0d_1d_0)(b_0d_1d_0)b_1c_1,$$

applied to all subwords of the form $b_0a_1d_0$ and $b_0c_1d_0$, we get a word which decomposes into the product of subwords from a smaler list:

$$a_1, \quad b_1, \quad c_1, \quad d_1, \quad b_0b_1d_0, \quad b_0d_1d_0.$$

We keep notation w for the obtained word. Let

$$w = y_1 \dots y_m,$$

where $y_1, \dots, y_m \in \{a_1, b_1, c_1, d_1, b_0b_1d_0, b_0d_1d_0\}$, be the corresponding decomposition.

Under all transforms discussed above the word w was remaining 1-balanced. This means that by replacing all the subwords of the form $a_1, d_1, b_0d_1d_0$ with a left bracket and all the subwords of the form $b_1, c_1, b_0c_1d_0$ with a right bracket, one gets a well balanced bracket string. This implies that there exist such $i_1, \dots, i_{m-1} \geq 0$ that all the words

$$
\begin{aligned}
x_1 &= y_1b_1^{i_1},\\
x_2 &= d_1^{i_1}y_2b_1^{i_2},\\
&\vdots\\
x_{m-1} &= d_1^{i_{m-2}}y_{m-1}b_1^{i_{m-1}},\\
x_m &= d_1^{i_{m-1}}y_m
\end{aligned}
$$

are 1-balanced. Additionally, the word $x_1 \dots x_m$ is equivalent to the word w modulo relations (29). Having removed all subwords of the form $d_1^kb_1^k$ in each x_j, we get words from the required set

$$\{d_1^{i-1}a_1b_1^i, d_1^ic_1b_1^{i-1}, d_1^{i-1}b_0d_1d_0b_1^i, d_1^ib_0b_1d_0b_1^{i-1}\}_{i>0}$$

and some number of words $\varnothing$. This completes the proof of Lemma 4. □

Now we return to the proof of Theorem 1. Let two words w_1, w_2 in the alphabet A correspond to isotopic tangles. We wish to show that they can be obtained from each other by applying relations (24)–(31). Clearly we have $\varphi(w_1) = \varphi(w_2)$. Also clear that one can find a word u, consisting only of the letters $\{b_i, d_i\}$, such that $\varphi(u)_i = -\varphi(w_1)_i$ for all $i \neq 0$. We have:

$$\varphi(w_1u)_i = \varphi(w_2u)_i = 0, \quad \text{for all } i \neq 0.$$

It easily follows from this that for a sufficiently large N the words

$$w'_j = d_1^{N^2} \dots d_{N-1}^{2N} d_N^N b_0^{N^2} \dots b_{2-N}^{2N} b_{1-N}^N w_j u d_{1-N}^N d_{2-N}^{2N} \dots d_0^{N^2} b_N^N b_{N-1}^{2N} \dots b_1^{N^2},$$

where $j = 1, 2$, are i-balanced for all $i \neq 0$.

Due to relations (29), the words w_1 and w_2 are equivalent modulo relations (24)–(31) if and only if the same is true about w'_1 and w'_2.

Thus it is enough to proof the assertion in the case when the words w_1 and w_2 are i-balanced for all $i \neq 0$. In this case, according to Lemma 4, by using relations (24)–(31) both words can be decomposed into the product of the generators (32)–(35) of the semigroup $T \in Y_\infty$. Therefore, it suffices to show that relations (2)–(18), which defines the semigroup T, follows from relations (24)–(31) under indentifications (32)–(35). We devote the next section to checking this.

4. Checking relations

So, here we shall check that relations (2)–(18) for the elements (32)–(35) are satisfied by virtue of (24)–(31).

Relation (2):

$$\begin{aligned}
\alpha_i \beta_{i+1} &= d_1^{i-1} a_1 b_1^i d_1^{i+1} c_1 b_1^i \\
&= d_1^{i-1} (a_1 d_1 c_1 b_1) b_1^{i-1} \\
&= d_1^{i-1} (a_1 (b_0 c_0) b_1) b_1^{i-1} \\
&= d_1^{i-1} (a_1 d_2 b_2 (b_0 c_0) b_1) b_1^{i-1} \\
&= d_1^{i-1} (a_1 d_2 (b_0 c_0) b_2 b_1) b_1^{i-1} \\
&= d_1^{i-1} (a_1 d_2 (d_1 c_1) b_2 b_1) b_1^{i-1} \\
&= d_1^{i-1} (a_1 b_1 b_0 (d_0 d_1 d_2) d_1 c_1 b_2 b_1) b_1^{i-1} \\
&= d_1^{i-1} (a_1 b_1 b_0 d_1 c_1 (d_0 d_1 d_2) b_2 b_1) b_1^{i-1} \\
&= d_1^{i-1} ((a_1 b_1) b_0 (d_1 c_1) d_0) b_1^{i-1} \\
&= d_1^{i-1} ((a_0 d_0) b_0 (b_0 c_0) d_0) b_1^{i-1} \\
&= d_1^{i-1} (a_0 b_0 c_0 d_0) b_1^{i-1} \\
&= d_1^{i-1} b_1^{i-1} \\
&= 1.
\end{aligned}$$

Note that simaltaniously we have proven the relation

$$b_i a_i d_i c_i = 1,$$

which is symmetric to (28).

Relation (3):

$$\begin{aligned}
\alpha_{i+1} \beta_i &= d_1^i a_1 b_1^{i+1} d_1^i c_1 b_1^{i-1} \\
&= d_1^i a_1 b_1 c_1 b_1^{i-1} \\
&= d_1^i (a_1 b_1 c_1 d_1) b_1^i \\
&= d_1^i b_1^i \\
&= 1.
\end{aligned}$$

Let us prove the following auxiliary relations:

$$d_1^{i-1} a_1 b_1^i = b_0^{i-1} a_0 d_0^i, \tag{38}$$

$$d_1^i c_1 b_1^{i-1} = b_0^i c_0 d_0^{i-1}, \tag{39}$$

$i > 0$. To this end, we apply induction. For $i = 1$ these relations coincide with (26), (27). For $i > 1$ we have:

$$\begin{aligned}
d_1^{i-1} a_1 b_1^i &= d_1 (d_1^{i-2} a_1 b_1^{i-1}) b_1 \\
&= d_1 (d_1^{i-2} a_1 b_1^{i-1}) b_1 \\
&= d_1 (b_0^{i-2} a_0 d_0^{i-1}) b_1 \\
&= b_0 b_{-1} (d_{-1} d_0 d_1)(b_0^{i-2} a_0 d_0^{i-1}) b_1 \\
&= b_0 b_{-1} (b_0^{i-2} a_0 d_0^{i-1})(d_{-1} d_0 d_1) b_1 \\
&= b_0 b_{-1} (b_0^{i-2} a_0 d_0^{i-1}) d_{-1} d_0 \\
&= b_0 b_{-1} (d_1^{i-2} a_1 b_1^{i-1}) d_{-1} d_0 \\
&= b_0 (d_1^{i-2} a_1 b_1^{i-1}) b_{-1} d_{-1} d_0 \\
&= b_0 (b_0^{i-2} a_0 d_0^{i-1}) d_0 \\
&= b_0^{i-1} a_0 d_0^i.
\end{aligned}$$

Similarly, one proves (39).

Let us prove (4).

$$\begin{aligned}
\alpha_i \alpha_{i+j+1} &= d_1^{i-1} a_1 b_1^i d_1^{i+j} a_1 b_1^{i+j+1} \\
&= d_1^{i-1} (a_1 d_1)(d_1^{j-1} a_1 b_1^j) b_1^{i+1} \\
&= d_1^{i-1} (a_2 b_2)(b_0^{j-1} a_0 d_0^j) b_1^{i+1} \\
&= d_1^{i-1} (b_0^{j-1} a_0 d_0^j)(a_2 b_2) b_1^{i+1} \\
&= d_1^{i-1} (d_1^{j-1} a_1 b_1^j)(a_1 d_1) b_1^{i+1} \\
&= d_1^{i+j-2} a_1 b_1^j a_1 b_1^i \\
&= (d_1^{i+j-2} a_1 b_1^{i+j-1})(d_1^{i-1} a_1 b_1^i) \\
&= \alpha_{i+j-1} \alpha_i.
\end{aligned}$$

Relation (5) is proven similarly, this time by using (39). As for (6), we have:

$$\begin{aligned}
\beta_i \beta_{i+j-1} &= d_1^i c_1 b_1^{i-1} d_1^{i+j-1} c_1 b_1^{i+j-2} \\
&= d_1^{i+1} (b_1 c_1)(d_1^j c_1 b_1^{j-1}) b_1^{i-1} \\
&= d_1^{i+1} (d_2 c_2)(b_0^j c_0 d_0^{j-1}) b_1^{i-1} \\
&= d_1^{i+1} (b_0^j c_0 d_0^{j-1})(d_2 c_2) b_1^{i-1} \\
&= d_1^{i+1} (d_1^j c_1 b_1^{j-1})(b_1 c_1) b_1^{i-1} \\
&= (d_1^{i+j+1} c_1 b_{i+j})(d_1^i c_1 b_1^{i-1}) \\
&= \beta_{i+j+1} \beta_i.
\end{aligned}$$

One proves (7) in a similar way.

In order to prove (8)–(12) we shall make use of the following relations (for $i > 0$):

$$b_0 d_1 d_0 b_1 = b_0^2 b_{-1} d_0 d_{-1} d_0 = d_1 d_2 d_1 b_2 b_1^2, \tag{40}$$

$$d_1^{i-1} (b_0 d_1 d_0 b_1) b_1^{i-1} = b_0^{i-1} (b_0 d_1 d_0 b_1) d_0^{i-1}. \tag{41}$$

Let us prove (40):

$$\begin{aligned}
b_0^2 b_{-1} d_0 d_{-1} d_0 &= b_0^2 b_{-1} d_0 (d_{-1} d_0 d_1) b_1 \\
&= b_0^2 b_{-1} (d_{-1} d_0 d_1) d_0 b_1 \\
&= b_0 d_1 d_0 b_1 \\
&= b_0 d_1 (d_0 d_1 d_2) b_2 b_1^2 \\
&= b_0 (d_0 d_1 d_2) d_1 b_2 b_1^2 \\
&= d_1 d_2 d_1 b_2 b_1^2.
\end{aligned}$$

We shall prove relation (41) by induction. For $i = 1$ it is trivial. Next, we have:

$$\begin{aligned}
d_1^{i-1}(b_0d_1d_0b_1)b_1^{i-1} &= d_1(d_1^{i-2}(b_0d_1d_0b_1)b_1^{i-2})b_1 \\
&= d_1(b_0^{i-2}(b_0d_1d_0b_1)d_0^{i-2})b_1 \\
&= d_1(b_0^{i-2}(b_0^2b_{-1}d_0d_{-1}d_0)d_0^{i-2})b_1 \\
&= b_0b_{-1}b_{-2}(d_{-2}d_{-1}d_0d_1)(b_0^{i-2}(b_0^2b_{-1}d_0d_{-1}d_0)d_0^{i-2})b_1 \\
&= b_0b_{-1}b_{-2}(b_0^{i-2}(b_0^2b_{-1}d_0d_{-1}d_0)d_0^{i-2})(d_{-2}d_{-1}d_0d_1)b_1 \\
&= b_0b_{-1}b_{-2}(b_0^{i-2}(b_0^2b_{-1}d_0d_{-1}d_0)d_0^{i-2})d_{-2}d_{-1}d_0 \\
&= b_0b_{-1}b_{-2}(d_1^{i-2}(b_0d_1d_0b_1)b_1^{i-2})d_{-2}d_{-1}d_0 \\
&= b_0b_{-1}(d_1^{i-2}(b_0d_1d_0b_1)b_1^{i-2})d_{-1}d_0 \\
&= b_0(d_1^{i-2}b_{-1}(b_0d_1d_0b_1)d_{-1}b_1^{i-2})d_0 \\
&= b_0(d_1^{i-2}b_{-1}(d_1d_2d_1b_2b_1^2)d_{-1}b_1^{i-2})d_0 \\
&= b_0(d_1^{i-2}(d_1d_2d_1b_2b_1^2)b_1^{i-2})d_0 \\
&= b_0(d_1^{i-2}(b_0d_1d_0b_1)b_1^{i-2})d_0 \\
&= b_0(b_0^{i-2}(b_0d_1d_0b_1)d_0^{i-2})d_0 \\
&= b_0^{i-1}(b_0d_1d_0b_1)d_0^{i-1}
\end{aligned}$$

Relation (8):

$$\begin{aligned}
\alpha_i\sigma_{i+j+1} &= (d_1^{i-1}a_1b_1^i)(d_1^{i+j}b_0d_1d_0b_1^{i+j+1}) \\
&= d_1^{i-1}(a_1d_1)(d_1^{j-1}(b_0d_1d_0b_1)b_1^{j-1})b_1^{i+1} \\
&= d_1^{i-1}(a_2b_2)(b_0^{j-1}(b_0d_1d_0b_1)d_0^{j-1})b_1^{i+1} \\
&= d_1^{i-1}(a_2b_2)(b_0^{j-1}(b_0^2b_{-1}d_0d_{-1}d_0)d_0^{j-1})b_1^{i+1} \\
&= d_1^{i-1}(b_0^{j-1}(b_0^2b_{-1}d_0d_{-1}d_0)d_0^{j-1})(a_2b_2)b_1^{i+1} \\
&= d_1^{i-1}(d_1^{j-1}(b_0d_1d_0b_1)b_1^{j-1})(a_1d_1)b_1^{i+1} \\
&= \sigma_{i+j-1}\alpha_i.
\end{aligned}$$

Relation (9):

$$\begin{aligned}
\alpha_{i+j+1}\sigma_i &= (d_1^{i+j}a_1b_1^{i+j+1})(d_1^{i-1}b_0d_1d_0b_1^i) \\
&= d_1^{i-1}(d_1^{j+2}a_1b_1^{j+2})b_0d_1d_0b_1^i \\
&= d_1^{i-1}(b_0^{j+1}a_0d_0^{j+2})b_0d_1d_0b_1^i \\
&= d_1^{i-1}b_0(b_0^ja_0d_0^{j+1})d_1d_0b_1^i \\
&= d_1^{i-1}b_0d_1(d_1^{j-1}a_1b_1^j)d_0b_1^i \\
&= d_1^{i-1}b_0d_1(b_0^{j-1}a_0d_0^j)d_0b_1^i \\
&= d_1^{i-1}b_0d_1d_0(b_0^ja_0d_0^{j+1})b_1^i \\
&= d_1^{i-1}b_0d_1d_0(d_1^ja_1b_1^{j+1})b_1^i \\
&= \sigma_i\alpha_{i+j+1}.
\end{aligned}$$

Relations (10) and (11) are proven in a similar way. Let us prove (12):

$$
\begin{aligned}
\sigma_i\sigma_{i+j+1} &= (d_1^{i-1}b_0d_1d_0b_1^i)(d_1^{i+j}b_0d_1d_0b_1^{i+j+1}) \\
&= d_1^{i-1}b_0d_1d_0(d_1^j(b_0d_1d_0b_1)b_1^j)b_1^i \\
&= d_1^{i-1}b_0d_1d_0(b_0^j(b_0d_1d_0b_1)d_0^j)b_1^i \\
&= d_1^{i-1}b_0d_1(b_0^{j-1}(b_0d_1d_0b_1)d_0^{j-1})d_0b_1^i \\
&= d_1^{i-1}b_0d_1(d_1^{j-1}(b_0d_1d_0b_1)b_1^{j-1})d_0b_1^i \\
&= d_1^{i-1}b_0(d_1^j(b_0d_1d_0b_1)b_1^j)d_1d_0b_1^i \\
&= d_1^{i-1}b_0(b_0^j(b_0d_1d_0b_1)d_0^j)d_1d_0b_1^i \\
&= d_1^{i-1}(b_0^{j+1}(b_0d_1d_0b_1)d_0^{j+1})b_0d_1d_0b_1^i \\
&= d_1^{i-1}(d_1^{j+1}(b_0d_1d_0b_1)b_1^{j+1})b_0d_1d_0b_1^i \\
&= \sigma_{i+j+1}\sigma_i.
\end{aligned}
$$

Relations (13):

$$
\begin{aligned}
\alpha_i\sigma_{i+1}\beta_{i+2} &= (d_1^{i-1}a_1b_1^i)(d_1^ib_0d_1d_0b_1^{i+1})(d_1^{i+2}c_1b_1^{i+1}) \\
&= d_1^{i-1}(a_1b_0d_1d_0d_1c_1b_1)b_1^{i-1} \\
&= d_1^{i-1}(a_1b_0d_1(d_0d_1d_2)b_2c_1b_1^2)b_1^{i-1} \\
&= d_1^{i-1}(a_1b_0(d_0d_1d_2)d_1b_2c_1b_1^2)b_1^{i-1} \\
&= d_1^{i-1}(a_1d_1d_2d_1b_2c_1b_1^2)b_1^{i-1} \\
&= d_1^{i-1}(a_2b_2d_2d_1b_2c_1b_1^2)b_1^{i-1} \\
&= d_1^{i-1}(a_2d_1b_2c_1b_1^2)b_1^{i-1} \\
&= d_1^{i-1}(a_2(d_1d_2d_3)b_3b_2^2c_1b_1^2)b_1^{i-1} \\
&= d_1^{i-1}((d_1d_2d_3)a_2b_3b_2^2c_1b_1^2)b_1^{i-1} \\
&= d_1^{i-1}(d_1b_0d_0(d_2d_3a_2b_3b_2^2)c_1b_1^2)b_1^{i-1} \\
&= d_1^{i-1}(d_1b_0(d_2d_3a_2b_3b_2^2)d_0c_1b_1^2)b_1^{i-1} \\
&= d_1^{i-1}(d_1b_0b_1(d_1d_2d_3)a_2b_3b_2^2d_0c_1b_1^2)b_1^{i-1} \\
&= d_1^{i-1}(d_1b_0b_1a_2(d_1d_2d_3)b_3b_2^2d_0c_1b_1^2)b_1^{i-1} \\
&= d_1^{i-1}(d_1b_0b_1a_2d_1b_2(d_0c_1)b_1^2)b_1^{i-1} \\
&= d_1^{i-1}(d_1b_0b_1(a_2b_2)b_1b_0(d_0d_1d_2)d_1b_2(d_0c_1)b_1^2)b_1^{i-1} \\
&= d_1^{i-1}(d_1b_0b_1(a_1d_1)b_1b_0d_1(d_0d_1d_2)b_2(b_1c_0)b_1^2)b_1^{i-1} \\
&= d_1^{i-1}(d_1b_0b_1a_1b_0d_1(d_0c_0)b_1^2)b_1^{i-1} \\
&= d_1^{i-1}(d_1b_0b_1a_1b_0d_1(b_{-1}c_{-1})b_1^2)b_1^{i-1} \\
&= d_1^{i-1}(d_1b_0b_1a_1b_0(b_{-1}c_{-1})d_1b_1^2)b_1^{i-1} \\
&= d_1^{i-1}(d_1b_0b_1a_1b_0(d_0c_0)b_1)b_1^{i-1} \\
&= d_1^{i-1}(d_1b_0b_1a_1c_0b_1)b_1^{i-1} \\
&= d_1^{i-1}(d_1b_0b_1(a_1b_1)b_0b_{-1}(d_{-1}d_0d_1)c_0b_1)b_1^{i-1} \\
&= d_1^{i-1}(d_1b_0b_1(a_0d_0)b_0b_{-1}c_0(d_{-1}d_0d_1)b_1)b_1^{i-1} \\
&= d_1^{i-1}(d_1b_0b_1(a_0b_{-1})c_0d_{-1}d_0)b_1^{i-1} \\
&= d_1^{i-1}(d_1b_0b_1a_{-1}(d_0c_0)d_{-1}d_0)b_1^{i-1} \\
&= d_1^{i-1}(d_1b_0b_1(a_{-1}b_{-1}c_{-1}d_{-1})d_0)b_1^{i-1} \\
&= d_1^{i-1}(d_1b_0b_1d_0)b_1^{i-1} \\
&= \sigma_i^{-1},
\end{aligned}
$$

$$
\begin{aligned}
\alpha_i\sigma_{i+1}^{-1}\beta_{i+2} &= (d_1^{i-1}a_1b_1^i)(d_1^{i+1}b_0b_1d_0b_1^i)(d_1^{i+2}c_1b_1^{i+1}\\
&= d_1^{i-1}((a_1d_1)b_0b_1d_0d_1(d_1c_1)b_1^2)b_1^{i-1}\\
&= d_1^{i-1}((a_2b_2)b_0b_1d_0d_1(b_0c_0)b_1^2)b_1^{i-1}\\
&= d_1^{i-1}(b_0(a_2b_2)b_1d_0d_1d_2b_2(b_0c_0)b_1^2)b_1^{i-1}\\
&= d_1^{i-1}(b_0(a_1d_1)b_1d_0d_1d_2(b_0c_0)b_2b_1^2)b_1^{i-1}\\
&= d_1^{i-1}(b_0a_1(d_0d_1d_2)(d_1c_1)b_2b_1^2)b_1^{i-1}\\
&= d_1^{i-1}(b_0a_1(d_1c_1)(d_0d_1d_2)b_2b_1^2)b_1^{i-1}\\
&= d_1^{i-1}(b_0(a_1d_1c_1b_1)d_1d_0b_1)b_1^{i-1}\\
&= d_1^{i-1}(b_0d_1d_0b_1)b_1^{i-1}\\
&= \sigma_i.
\end{aligned}
$$

Relations (14) are symmetric to relations (13), their proofs can be received from the proofs of relations (13) by reversing the letter order in all the words and replacings $a_i \leftrightarrow c_i$, $b_i \leftrightarrow d_i$.

Relations (15) are symmetric to each other. Let us prove one of them:

$$
\begin{aligned}
\alpha_i\sigma_{i+1}\beta_i &= (d_1^{i-1}a_1b_1^i)(d_1^ib_0d_1d_0b_1^{i+1})(d_1^ic_1b_1^{i-1})\\
&= d_1^{i-1}(a_1b_0d_1d_0(b_1c_1))b_1^{i-1}\\
&= d_1^{i-1}(a_1b_0d_1d_0(d_2c_2)b_1^{i-1}\\
&= d_1^{i-1}(a_1b_0d_1(b_1c_1)d_0)b_1^{i-1}\\
&= d_1^{i-1}((a_1b_0)c_1d_0)b_1^{i-1}\\
&= d_1^{i-1}(a_0(d_1c_1)d_0)b_1^{i-1}\\
&= d_1^{i-1}(a_0(b_0c_0)d_0)b_1^{i-1}\\
&= 1.
\end{aligned}
$$

Relations (16) also are symmetric to each other. We prove one of them:

$$
\begin{aligned}
\alpha_{i+1}\sigma_i\beta_{i+1} &= (d_1^ia_1b_1^{i+1})(d_1^{i-1}b_0d_1d_0b_1^i)(d_1^{i+1}c_1b_1^i)\\
&= d_1^i((a_1b_1)b_1b_0d_1d_0d_1c_1)b_1^i\\
&= d_1^i((a_0d_0)b_1b_0d_1d_0d_1c_1)b_1^i\\
&= d_1^i(d_2(a_0d_0)(b_2b_1b_0)d_1d_0d_1c_1)b_1^i\\
&= d_1^i(d_2(a_1b_1)(b_2b_1b_0)d_1d_0d_1c_1)b_1^i\\
&= d_1^i(d_2(b_2b_1b_0)(a_1b_1)d_1d_0d_1c_1)b_1^i\\
&= d_1^i(b_1b_0a_1(d_0d_1d_2)b_2c_1)b_1^i\\
&= d_1^i(b_1b_0(d_0d_1d_2)a_1b_2c_1)b_1^i\\
&= d_1^i(d_2a_1(b_2c_1))b_1^i\\
&= d_1^i(d_2(a_1d_1)c_2)b_1^i\\
&= d_1^i(d_2(a_2b_2)c_2)b_1^i\\
&= 1.
\end{aligned}
$$

Relations (17) trivially follows from (29). It remains to check relation (18):

$$\begin{aligned}
\sigma_i\sigma_{i+1}\sigma_i &= (d_1^{i-1}b_0d_1d_0b_1^i)(d_1^ib_0d_1d_0b_1^{i+1})(d_1^{i-1}b_0d_1d_0b_1^i)\\
&= d_1^{i-1}b_0d_1^2d_0b_1^2b_0d_1d_0b_1^i\\
&= d_1^{i-1}b_0d_1b_0b_{-1}(d_{-1}d_0d_1)d_0b_1^2b_0d_1d_0b_1^i\\
&= d_1^{i-1}b_0d_1b_0b_{-1}d_0(d_{-1}d_0d_1)b_1^2b_0d_1d_0b_1^i\\
&= d_1^{i-1}b_0d_1(d_2b_2)(b_0b_{-1}d_0d_{-1}d_0)b_1b_0d_1d_0b_1^i\\
&= d_1^{i-1}b_0d_1d_2(b_0b_{-1}d_0d_{-1}d_0)b_2b_1b_0d_1d_0b_1^i\\
&= d_1^{i-1}b_0d_1d_2b_0b_{-1}d_0(d_{-1}d_0d_1)b_1b_2b_1b_0d_1d_0b_1^i\\
&= d_1^{i-1}b_0d_1d_2b_0b_{-1}(d_{-1}d_0d_1)d_0b_1b_2b_1b_0d_1d_0b_1^i\\
&= d_1^{i-1}b_0^2b_{-1}(d_{-1}d_0d_1d_2)(d_1d_0b_1)b_2b_1b_0d_1d_0b_1^i\\
&= d_1^{i-1}b_0^2b_{-1}(d_1d_0b_1)(d_{-1}d_0d_1d_2)b_2b_1b_0d_1d_0b_1^i\\
&= d_1^{i-1}b_0^2b_{-1}d_1d_0b_1d_{-1}d_1d_0b_1^i\\
&= d_1^{i-1}b_0^2b_{-1}d_1d_0b_1d_1d_{-1}d_0b_1^i\\
&= d_1^{i-1}b_0^2b_{-1}d_1d_0(d_{-1}d_0d_1)b_1^{i+1}\\
&= d_1^{i-1}b_0^2b_{-1}d_1(d_{-1}d_0d_1)d_0b_1^{i+1}\\
&= d_1^{i-1}b_0^2d_1d_0d_1d_0b_1^{i+1}\\
&= d_1^{i-1}b_0^2d_1(d_0d_1d_2)b_2d_0b_1^{i+1}\\
&= d_1^{i-1}b_0^2(d_0d_1d_2)d_1b_2d_0b_1^{i+1}\\
&= d_1^{i-1}b_0d_1d_2d_1b_2(b_{-1}d_{-1})d_0b_1^{i+1}\\
&= d_1^{i-1}b_0b_{-1}d_1d_2d_1b_2d_{-1}d_0b_1^{i+1}\\
&= d_1^{i-1}b_0b_{-1}b_0(d_0d_1d_2)d_1b_2d_{-1}d_0b_1^{i+1}\\
&= d_1^{i-1}b_0b_{-1}b_0d_1(d_0d_1d_2)b_2d_{-1}d_0b_1^{i+1}\\
&= d_1^{i-1}b_0b_{-1}b_0d_1d_0d_1d_{-1}d_0b_1^{i+1}\\
&= d_1^{i-1}b_0b_{-1}b_0d_1d_0d_{-1}d_1d_0b_1^{i+1}\\
&= d_1^{i-1}b_0b_{-1}(b_0d_1d_0)(d_{-1}d_0d_1d_2)b_2b_1b_0d_1d_0b_1^{i+1}\\
&= d_1^{i-1}b_0b_{-1}(d_{-1}d_0d_1d_2)(b_0d_1d_0)b_2b_1b_0d_1d_0b_1^{i+1}\\
&= d_1^id_2b_0d_1d_0b_2b_1b_0d_1d_0b_1^{i+1}\\
&= d_1^id_2b_0^2b_{-1}(d_{-1}d_0d_1)d_0b_2b_1b_0d_1d_0b_1^{i+1}\\
&= d_1^id_2b_0^2b_{-1}d_0(d_{-1}d_0d_1)(b_2b_1b_0)d_1d_0b_1^{i+1}\\
&= d_1^id_2(b_0^2b_{-1}d_0d_{-1}d_0)(b_2b_1b_0)d_1^2d_0b_1^{i+1}\\
&= d_1^i(b_0^2b_{-1}d_0d_{-1}d_0)b_1b_0d_1^2d_0b_1^{i+1}\\
&= d_1^ib_0^2b_{-1}d_0(d_{-1}d_0d_1)b_1^2b_0d_1^2d_0b_1^{i+1}\\
&= d_1^ib_0^2b_{-1}(d_{-1}d_0d_1)d_0b_1^2b_0d_1^2d_0b_1^{i+1}\\
&= (d_1^ib_0d_1d_0b_1^{i+1})(d_1^{i-1}b_0d_1d_0b_1^i)(d_1^ib_0d_1d_0b_1^{i+1})\\
&= \sigma_{i+1}\sigma_i\sigma_{i+1}.
\end{aligned}$$

5. Semigroups Y_n and groups D_n in the case $n < \infty$.

In Section 3 we defined subsemigroups $Y_n \subset Y_\infty$, where $n = 3, 4, \dots$. Clearly for any $n \geq 3$ the semigroup Y_n coincides with the semigroup in Y_∞ generated by the set $A_1 \cup A_2 \cup \dots \cup A_{n-1}$, where $A_i = \{a_i, b_i, c_i, d_i\}$.

Theorem 2 *For $n \geq 7$ the semigroup Y_n is isomorphic to the semigroup defined by the generating set $A_1 \cup A_2 \cup \cdots \cup A_{n-1}$ and those relations (24)–(31) that involve only generators from that set.*

PROOF. Clearly the semigroup Y_n is isomorphic to the semigroup Y_n' in Y_∞ generated by the set $A_{-2} \cup A_{-1} \cup \cdots \cup A_{n-5} \cup A_{n-4}$. The proof of our assertion for the semigroup Y_n' with $n \geq 7$ almost verbally coincides with the proof of Theorem 1. Indeed, the elements (32)–(35) belong to Y_n'. All the transforms that were used in the proof of Lemma 4 preserve the set of words in the alphabet $A_{-2} \cup A_{-1} \cup \cdots \cup A_{n-5} \cup A_{n-4}$ for $n \geq 7$. In the case of a finite n, one can simply omit all the letters b_i, d_i with $i > n-4$ and $i < -2$ in the definition of the words $w_{1,2}'$ in the end of Section 3. Finally, during the checking relations in Section 4, only generators from $A_{-2} \cup \cdots \cup A_3$ appeared. $\square$

For $5 \leq n < \infty$ the semigroups Y_n can be defined in a different way. By adding to the generating set the tangles of complexity 1 acting nontrivially in P_{n-1} and P_0, we get the following presentation.

Theorem 3 *If $n \geq 5$ the semigroup Y_n is isomorphic to the semigroup presented by the generating set $\{a_i, b_i, c_i, d_i\}_{i \in \mathbb{Z}_n}$ and the defining relations*

$$d_1 d_2 \ldots d_n = 1, \tag{42}$$

$$a_i d_{i+1} = a_{i+1} b_i, \tag{24}$$

$$d_i c_{i+1} = b_{i+1} c_i, \tag{25}$$

$$a_i d_i = a_{i+1} b_{i+1}, \tag{26}$$

$$b_i c_i = d_{i+1} c_{i+1}, \tag{27}$$

$$a_i b_i c_i d_i = 1, \tag{28}$$

$$b_i d_i = d_i b_i = 1, \tag{29}$$

$$x_i y_j = y_j x_i, \tag{30}$$

for all $i, j \in \mathbb{Z}_n$ such that $i - j \notin \{-1, 0, 1\}$ and $x_i \in \{a_i, b_i, c_i, d_i\}$, $y_j \in \{a_j, b_j, c_j, d_j\}$. All the indices are regarded modulo n.

Again, the proof is almost the same as that of Theorem 1. Here the generators a_i, b_i, c_i, d_i with $1 \leq 1 \leq n-1$ are the same as above. The new generators a_n, b_n, c_n, d_n acting in the half-planes P_{n-1} and P_0 are expressed in terms of the otheres due to the relations given above:

$$\begin{aligned}
a_n &= a_1 d_2 \ldots d_{n-1}, \\
b_n &= d_1 \ldots d_{n-1}, \\
c_n &= d_1 \ldots d_{n-2} c_{n-1}, \\
d_n &= b_{n-1} \ldots b_1.
\end{aligned}$$

Relations (31), which are omitted this time, easily follow from (30) and (42).

If $n = 3, 4$ the proof fails because it uses comutativity of the elements from A_i with the elements from A_{i+2} and from A_{i+3}, which does not take place in the case $n = 3, 4$. However, the semigroups Y_3 and Y_4 are also finitely presented. For Y_3 this fact was established in papers [3], [4]. For Y_4 the proof is similar. We give here only the formulation of the corresponding claim.

Theorem 4 *The semigroup Y_3 is isomorphic to the semigroup presented by the genertors $\{a_i, b_i, c_i, d_i\}_{i \in \mathbb{Z}_3}$ and the defining relations*

$$a_i = a_{i+1} d_{i-1}, \quad b_i = a_{i-1} c_{i+1}, \quad c_i = b_{i-1} c_{i+1}, \quad d_i = a_{i+1} c_{i-1},$$
$$d_1 d_2 d_3 = 1,$$
$$b_i d_i = d_i b_i = 1,$$
$$x_i y_i = y_i x_i,$$

where $x_i \in \{a_{i-1} b_{i-1},\ d_{i-1} c_{i-1},\ b_{i+1} d_{i-1} d_{i+1} b_{i-1}\}$, $y_i \in \{a_i,\ b_i,\ c_i,\ b_{i-1} d_i d_{i-1}\}$, $i \in \mathbb{Z}_3$.

The semigroup Y_4 is isomorphic to the semigroup presented by the generators $\{a_i, b_i, c_i, d_i\}_{i\in\mathbb{Z}_4}$ and the defining reltions

$$\begin{aligned} d_1d_2d_3d_4 &= 1, \\ a_id_{i+1} &= a_{i+1}b_i, \\ d_ic_{i+1} &= b_{i+1}c_i, \\ a_id_i &= a_{i+1}b_{i+1}, \\ b_ic_i &= d_{i+1}c_{i+1}, \\ a_ib_ic_id_i &= 1, \\ b_id_i = d_ib_i &= 1, \\ x_iy_{i+2} &= y_{i+2}x_i, \end{aligned}$$

where $x_i \in \{a_i, b_i, c_i, d_i\}$, $y_i \in \{a_i, b_i, c_i, d_i, b_{i-1}a_id_{i-1}, b_{i-1}b_id_{i-1}, b_{i-1}c_id_{i-1}\}$, $i \in \mathbb{Z}_4$.

Theorem 5 *If $n \geq 3$ the center $Z(Y_n)$ of the semigroup Y_n is isomorphic to the semigroup L of isotopy classes of non-oriented links in $\mathbb{R}^3$.*

PROOF. We say that a word w is *balanced* if it is balanced for all $i \in \mathbb{Z}$. Clearly the semigroup L is embedded Y_∞ as the subsemigroup consisting of all the elements that *can* be presented by a balanced word. Also clear that this subsemigroup is contained in all the semigroups $T, Y_3, Y_4, \dots$ and that all its elements commute with all elements from Y_∞. Let us prove the converse statement, i.e. if a word w presents a central element x of the semigroup Y_n, then x belongs to the semigroup $L \subset Y_n \subset Y_\infty$.

Indeed, it is not hard to show that if $x \in Z(Y_n)$, then $\varphi(x) = 0$. Hence for some N and $i_1, i_2, \dots, i_N \in \{1, 2, \dots, n-1\}$ the word $a_{i_1}\dots a_{i_N}wc_{i_N}\dots c_{i_1}$ is balanced. Then for the corresponding tangle t, there exists a sphere embedded in $\mathbb{R}^3$ that separates the tangle into a link and a tangle presenting the unit of the semigroup Y_n. Therefore, such a sphere exists also for an equivalent tangle defined by the word $a_{i_1}\dots a_{i_N}c_{i_N}\dots c_{i_1}w$, and hence for the tangle defined by the word w, since the word $a_{i_1}\dots a_{i_N}c_{i_N}\dots c_{i_1}$ is balanced. □

Consider now the group D_n of invertible elements of the semigroup Y_n, $n \geq 3$.

Theorem 6 *If $n \geq 3$ the group D_n is finitely presented and it can be presented by the generators $\{d_i\}_{i\in\mathbb{Z}_n}$ and the following set of defining relations:*

for $n \geq 5$

$$\begin{aligned} d_id_j &= d_jd_i, \quad \text{if } i-j \neq \pm 1, \\ d_1d_2\dots d_n &= 1, \end{aligned}$$

where all the indeces are regarded modulo n;

for $n = 4$

$$\begin{aligned} d_1d_2d_3d_4 &= 1, \\ d_id_{i+2} &= d_{i+2}d_i, \\ d_{i+2}(d_{i-1}^{-1}d_id_{i-1}) &= (d_{i-1}^{-1}d_id_{i-1})d_{i+2}, \end{aligned}$$

$i \in \mathbb{Z}_4$;

for $n = 3$

$$\begin{aligned} d_1d_2d_3 &= 1, \\ d_iu_i &= u_id_i, \\ u_iu_{i+1} &= u_{i+1}u_i, \end{aligned}$$

where $u_i = d_{i+1}^{-1}d_{i-1}d_{i+1}d_{i-1}^{-1}$, $i \in \mathbb{Z}_3$.

The commutator subgroup of the group D_n is isomorphic to the braid group B_∞ on the infinite number of strands.

PROOF. It is known that the group of invertible elements of the semgigroup T is the braid group B_∞, which is presented by generators σ_i (fig. 2) and relation (12) and (18). The proof of the first assertion of the theorem can be obtained from the proofs of similar statements for the semigroups Y_n, $n = 3, 4, \dots, \infty$, by replacing T with B_∞ and deleting all relations and calculations that use the generators a_i, c_i of the semigroups Y_n and the generators α_i, β_i of the semigroup T. In particular, in this way we conclude that the group of the invertible elements of the semigroup Y_∞ is presented by the generatros d_i, $i \in \mathbb{Z}$ and the relations:

$$d_i d_j = d_j d_i, \quad \text{if } |i-j| > 1$$
$$d_i(d_{i-1}d_i d_{i+1}) = (d_{i-1}d_i d_{i+1})d_i.$$

Also clear that the kernel of the restriction of the homomorphism φ to the group D_n contains the commutator subgroup of the group D_n and at the same time, it is a subgroup isomorphic to the braid group B_∞. So, it suffices to show that the commutator subgroup is the whole subgroup, not a non-trivial part of it. This follows easily from the fact that the generators (34), which twist a coule of strands, lie in the commutator subgroup of D_∞. $\square$

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