Toral rank conjecture for moment-angle complexes

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Abstract

In this paper we introduce an operation on the set of simplicial complexes, which we shall call "doubling operation". We show that the moment-angle complex $\mathbb{Z}K$ is the real moment-angle complex $\mathbb{R}L(K)$ for simplicial complex $L(K)$ obtained from $K$ by applying "doubling operation". As an application of this operation we prove the toral rank conjecture for $\mathbb{Z}K$ by estimating the lower bound of the cohomology rank (with rational coefficients) of the real moment-angle complexes $\mathbb{R}Z$. In, [4] the "doubling operation" for the polytopes was defined and the same result was proved for the class of the moment-angle manifolds, so this article can be considered as the extension of the previous one.

1 Doubling operations

Here we give the definition of the "doubling operation" and discuss its main properties.

Definition 1.1. Let $K$ be an arbitrary simplicial complex on the vertex set $[m] = \{v_1, \ldots, v_m\}$. The double of $K$ is the simplicial complex $L(K)$ on the vertex set $[2m] = \{v_1, v'_1, \ldots, v_m, v'_m\}$ determined by the following condition: $\omega \subset [2m]$ is the minimal (by inclusion) missing simplex of $L(K)$ iff $\omega$ is of the form $\{v_{i_1}, v'_{i_1}, \ldots, v_{i_k}, v'_{i_k}\}$, where $\{v_{i_1}, \ldots, v_{i_k}\}$ is a missing simplex of $K$.

If $K = \partial P^*$ is a boundary of the dual of the simple polytope $P$, then $L(K)$ coincides with $L(P)^*$, see the definition 1 in [4].

Examples.

• If $K = \Delta^m$ is the $(m-1)$-dimensional simplex, then $L(K) = \Delta^{2m}$.
• If $K = \partial \Delta^m$ is the boundary of the $(m-1)$-dimensional simplex, then $L(K) = \partial \Delta^{2m}$.

It is easy to see that "doubling operation" respects join of the simplicial complexes i.e. $L(K_1 \ast K_2) = L(K_1) \ast L(K_2)$.

Given a simplicial complex $K$ we denote by $\text{mdim} K$ the minimal dimension of the maximal by inclusion simplices. Thus, for any $K$ $\text{mdim} K \leq \dim K$, and $K$ is pure iff $\text{mdim} K = \dim K$.

The following lemma is the direct corollary from the definitions.

Lemma 1.2. Let $K$ be a simplicial complex on $[m]$, then $\dim L(K) = m + \dim K$ and $\text{mdim} L(K) = m + \text{mdim} K$.

2 $K$-powers

Definition 2.1. Let $(X, A)$ be a pair of CW — complexes. For a subset $\omega \subset [m]$ we define

$$(X, A)^\omega := \{(x_1, \ldots, x_m) \in X^m | x_i \in A \text{ for } i \notin \omega\}.$$
Now let $K$ be a simplicial complex on $[m]$. The $K$-power of the pair $(X, A)$ is

$$(X, A)^K := \bigcup_{\omega \in K} (X, A)^\omega.$$

In this paper we shall consider two examples of $K$-powers (see [1]):

- Moment-angle complexes $Z_K = (D^2, S^1)^K$.
- Real moment-angle complexes $\mathbb{R}Z_K = (I^2, S^0)^K$.

The next lemma explains the usefulness of the notion of “doubling operation” in the studying of the relationship between moment-angle complexes and real moment-angle complexes.

**Lemma 2.2.** Let $(X, A)$ be a pair of CW — complexes and $K$ be a simplicial complex on the vertex set $[m]$. Consider a pair $(Y, B) = (X \times X, (X \times A) \cup (A \times X))$. For this pair we have:

$$(Y, B)^K = (X, A)^{L(K)}.$$

In particular $Z_K = \mathbb{R}Z_{L(K)}$.

**Proof.** For a point $y = (y_1, \ldots, y_m) \in Y^m$ we set

$$\omega_Y(y) = \{v_i \in [m] \mid y_i \in Y \setminus B\} \subset [m].$$

For a point $x = (x_1, x'_1, \ldots, x_m, x'_m) \in X^{2m}$ the subset $\omega_X(x) \subset [2m]$ is defined in a similar way. Let $y = (y_1, \ldots, y_m) = ((x_1, x'_1), \ldots, (x_m, x'_m)) \in Y^m = X^{2m}$. It follows from the definition of the $K$-powers that $y \not\in (Y, B)^K$ iff $\omega_Y(y) \not\in K$. The latter is equivalent to the condition $\omega_X(x) \not\in L(K)$, where $x = (x_1, x'_1, \ldots, x_m, x'_m)$, since if $\omega_Y(y) = \{v_{i_1}, \ldots, v_{i_k}\}$ then $\omega_X(x) = \{v_{i_1}, v'_{i_1}, \ldots, v_{i_k}, v'_{i_k}\}$. Therefore

$$y \not\in (Y, B)^K \Leftrightarrow x \not\in (X, A)^{L(K)}$$

and the statement of the lemma is proved. □

**Example.** Let $K = \partial \Delta^2$ be the boundary of 1-simplex. Then we get decomposition of 3-dimensional sphere:

$$Z_K = D^2 \times S^1 \cup S^1 \times D^2 = S^3.$$

On the other hand $L(K) = \partial \Delta^4$ and $\mathbb{R}Z_{L(K)} = \partial I^4 = S^3$ is the boundary of the standard 4-dimensional cube. So, in accordance with the lemma, $Z_K = \mathbb{R}Z_{L(K)}$.

### 3 Toral rank conjecture

Let $X$ be a finite-dimensional topological space. Denote by $\text{trk}(X)$ the largest integer for which $X$ admits an almost free $T^{\text{trk}(X)}$ action.

**Conjecture** (Halperin’s toral rank conjecture, [3]),

$$\text{hrk}(X, Q) := \sum \dim H^i(X, Q) \geq 2^{\text{trk}(X)}.$$

Moment-angle complexes provide a big class of spaces with torus action, since there is natural coordinatewise $T^m$ action on the space $Z_K$. In fact for some $r$ one can choose subtorus $T^r \subset T^m$ such that the action $T^r$ on $Z_K$ is almost free. Our aim is to estimate the maximal rank of such subtorus and the lower bound of $\text{hrk}(Z_K, Q)$.

**Lemma 3.1.** Let $K$ be $(n - 1)$-dimensional simplicial complex on the vertex set $[m]$. Then the rank of subtorus $T^r \subset T^m$ that acts almost freely on $Z_K$ is less or equal to $m - n$. 

2
Proof. For a subset \( \omega \subset [m] \) we set \( T^\omega = (T, e)^\omega \) (see definition of \( K \)-powers), where \( e \in T \) is identity. It is easy to see that isotropy subgroups of the action \( T^m : Z_K \) are of the form \( T^\omega, \omega \in K \). Therefore \( T^\omega \subset T^m \) acts almost freely iff the set \( T^\omega \cap T^- \) is finite for any \( \omega \in K \).

Let \( \sigma \) be the simplex of the dimension \((n - 1)\). Since the intersection \( T^\sigma \cap T^- \) of two subtori in \( T^m \) is finite,

\[
\text{rk } T^\sigma + \text{rk } T^- \leq \text{rk } T^m,
\]

thus \( r \leq m - n \).

\( \Box \)

Remark. In fact for any \((n - 1)\)-dimensional complex \( K \) there is subtorus \( T^\sigma \subset T^m \) of the rank \( r = m - n \) that acts on \( Z_K \) almost freely, \( \cite{2} \) 7.1.

Now we prove our main result about the cohomology rank of the real moment-angle complexes.

**Theorem 3.2.** Let \( K \) be a simplicial complex on the vertex set \([m]\) with \( \text{mdim } K = n - 1 \). Then

\[
\text{hrk}(\mathbb{R}Z_K, \mathbb{Q}) \geq 2^{m-n}.
\]

We first formulate one general lemma.

**Lemma 3.3.** Let \((X, A)\) be a pair of \( CW \)-complexes; let \( U(A) \) be a neighbourhood of \( A \) in \( X \) such that \( (U(A), A) \simeq (A \times [0; 1], A \times \{0\}) \). Then the cohomology rank of the space \( Y \), obtained from the two copies of \( X \) by gluing them together along \( A \), \( Y = X_1 \cup_A X_2 \), satisfies inequality:

\[
\text{hrk}(Y, \mathbb{Q}) \geq \text{hrk}(X, \mathbb{Q}).
\]

This fact is direct consequence of the Mayer Vietoris long exact sequence.

**Proof of the theorem.** We shall prove this fact by induction on \( m \). The base of induction is trivial.

Assume this statement is true for the complexes with less than \( m \) vertices and \( K \) is the complex with \( m \) vertices.

The real moment-angle complex is a subspace of the \( m \)-dimensional cube \( \mathbb{R}Z_K \subset [-1; 1]^m \). Denote by \((x_1, \ldots, x_m)\) coordinates in \([-1; 1]^m\). Assume that the vertex \( v_1 \) belong to the maximal (by inclusion) simplex of \( K \) of the dimension \( \text{mdim } K = n - 1 \). Consider the decomposition of \( \mathbb{R}Z_K = M_+ \cup_X M_- \), where

\[
M_+ = \{ \vec{x} \in \mathbb{R}Z_K \subset \mathbb{R}^m \mid x_1 > 0 \},
\]

\[
M_- = \{ \vec{x} \in \mathbb{R}Z_K \subset \mathbb{R}^m \mid x_1 \leq 0 \},
\]

\[
X = \{ \vec{x} \in \mathbb{R}Z_K \subset \mathbb{R}^m \mid x_1 = 0 \}.
\]

It is easy to see that the pair \((M_+, X)\) satisfies the hypothesis of the lemma 3.3, so

\[
\text{hrk}(\mathbb{R}Z_K, \mathbb{Q}) \geq \text{hrk}(X, \mathbb{Q}).
\]

Now let’s describe the space \( X \) more explicitly. Let \( k \) be the number of vertices in the complex \( \text{lk } v_1 \). Then \( X \) is just the disjoint union of the \( 2^{m-k-1} \) copies of the space \( \mathbb{R}Z_{\text{lk } v_1} \). Moreover, since \( v_1 \) is vertex of the maximal (by inclusion) simplex of the minimal dimension \( n \), so \( \text{mdim } \text{lk } v_1 = n - 2 \). Thus, by the hypothesis of induction

\[
\text{hrk}(X, \mathbb{Q}) = 2^{m-k-1} \text{hrk}(\mathbb{R}Z_{\text{lk } v_1}, \mathbb{Q}) \geq 2^{m-k-1} \cdot 2^{k-(n-1)} = 2^{m-n}.
\]

The step of induction is proved.

Now let’s turn our attention to the moment-angle complexes. Combining the results of lemma 1.2, lemma 2.2 and theorem 3.2 we have:

\[
\text{hrk}(Z_K, \mathbb{Q}) = \text{hrk}(\mathbb{R}Z_{L(K)}) \geq 2^{2m - \text{mdim } L(K) - 1} = 2^{m - \text{mdim } K - 1} \geq 2^{m - \text{dim } K - 1}.
\]
Thus the toral rank conjecture holds for the action of subtori of $T^m$ on the moment-angle complexes $Z_K$.

The cohomology ring of $Z_K$ was calculated in [1]. One of the corollaries of this computation and Hochster’s theorem states (see [1], theorem 8.7):

**Theorem 3.4.**

$$H^*(Z_K, \mathbb{Z}) \cong \bigoplus_{\omega \subset [m], \ p \geq -1} \check{H}^p(K_\omega, \mathbb{Z}),$$

where $K_\omega$ is the restriction of $K$ on the subset $\omega \subset [m]$.

In view of this theorem we can reformulate our main result as follows:

$$\dim \bigoplus_{\omega \subset [m]} \check{H}^*(K_\omega, \mathbb{Q}) \geq 2^{m-n},$$

for any simplicial complex $K$ on $[m]$ with $\text{mdim } K = n - 1$.

The author is grateful to V. M. Buchstaber and scientific adviser T. E. Panov for suggesting the problem and attention to the research.

**References**


