BIGRADED BETTI NUMBERS OF CERTAIN SIMPLE POLYTOPES

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Abstract. There is a well-known in toric topology construction which associates a smooth manifold with a combinatorial simple polytope. Topological and analytic properties of these toric varieties, called moment-angle-manifolds, are in the area of main interest of toric topology and geometry. In this work we consider the manifolds, which arise from the parametried series of the so called Stasheff polytopes and stacked polytopes. we compute bigraded Betti numbers in the case of Stasheff polytopes K_{n+2} with $n \leq 5$ and the stacked polytopes of arbitrary dimension and an arbitrary amount of cut vertices. The bigraded Betti numbers in our examples are computed with the help of computer program Macaulay2, a brief description of the method is given below.

1. Introduction

In the first section we are going to introduce the notions of the so called face rings and bigraded Betti numbers of moment-angle-complexes that will be used during the whole work. The second section is dedicated to the case of the so called Stasheff polytopes or associahedra. The third one is dedicated to the stacked polytopes, which also gives us a series of simple polytopes. The author is grateful to Taras Panov for fruitful discussions and advice which was always so kindly proposed during all this work.

2. Stasheff polytopes

In the following we will denote by n the dimension, by m - the number of facets (vertices) of the simple (symplicial) polytopes. A few combinatorial ways to define the Stasheff polytope K_{n+2} are known. For details, see [3, Lecture II]. We will point out the following

Theorem 2.1. *(see* [3, Lecture II, Theorem 5.1]*).*
There exists an embedding

$$J : K_{n+1} \to \mathbb{R}^{n-1}$$

with an image

$$y = (y_1, \cdots, y_{n-1}) : 0 \leq y_l \leq l \cdot (n-l), y_i - y_{i+l} \leq i \cdot l,$$

where $l = 1, \cdots, n-1, i = 1, \cdots, n-l-1.$

If one considers

$L_{i,l} = \{y \in \mathbb{R}^{n-1} : y_i - y_{i+l} \leq i \cdot l, 1 \leq l \leq n-1, 0 \leq i \leq n-l, y_0 = y_n = 0\}.$

Key words and phrases. Betti numbers, moment-angle-complexes, Stasheff polytopes, stacked polytopes.

than our description gives us the image of K_{n+1} as an intersection of a $(n-1)$-cube with all the sets $L_{i,l}$. The proposition [3, Lecture II, Cor 6.2] allows us to reduce the computation of bigraded Betti numbers to the problem of finding the number of all noninteresecting pairs of diagonales in a corresponding polygon. In the examples below bigraded Betti numbers are got as a result of a program work, which is based on the following commands in the Macaulay2 computer stuff:

$$kk = ZZ/32749,$$

$$ringP = kk[v_1, \cdots, v_m],$$

$$I = ideal(\cdots),$$

$$betti\, res(ringP^1/I).$$

in the first line we generate the field of coefficients, in the third, in brackets, the monomial Stanley-Raisner ideal, consisting of elements of the form $v_i \cdot v_j$ is written, the res-command in the last line builds a minimal resolvent, and the betti-command gives us the cardinalities of the corresponding minimal bases.

In what follows the result of the described program work when $n \leq 5$ is given. For the convenience of the further observations, we will write the numbers appear in sets of $(n-1)$ vectors, which elements are the $\beta^{-i,2j}, i = 1, \cdots, m-n-1$, where $j - i = 1, \cdots, n-1$. Here we assume that $\beta^{0,0} = \beta^{-(m-n),2m} = 1$.

Example 2.2. (1) $n = 2, m = 5$

$$(5, 5)$$

(2) $n = 3, m = 9$

$$(15, 35, 24, 3, 0)$$

$$(0, 3, 24, 35, 15)$$

(3) $n = 4, m = 14$

$$(35, 140, 217, 154, 49, 7, 0, 0, 0)$$

$$(0, 28, 266, 784, 1094, 784, 266, 28, 0)$$

$$(0, 0, 0, 7, 49, 154, 217, 140, 35)$$

(4) $n = 5, m = 20$

$$(70, 420, 1089, 1544, 1300, 680, 226, 44, 4, 0, 0, 0, 0, 0)$$

$$(0, 144, 1796, 8332, 20924, 32309, 32184, 20798, 8480, 2053, 264, 12, 0, 0)$$

$$(0, 0, 12, 264, 2053, 8480, 20798, 32184, 32309, 20924, 8332, 1796, 144, 0)$$

$$(0, 0, 0, 0, 0, 4, 44, 226, 680, 1300, 1544, 1089, 420, 70)$$

3. Stacked polytopes

We are now to review the following

Definition. Lets call a **stacked polytope**$P(n,k)$ a simple polytope, which appears from a n-symplex by a consequtive cut of k vertices.

In our previous notation we immediately have $k=m-n-1$.
The dual symplicial polytope to the $P(n,k)$ is called a **pyramid suspension polytope** $Q(n,k)$. It can be obtained from the n-symplex by a consequtive suspension of cones above its facets. It is easy to convince oneself that for $k=1$ and $k=2$ the combinatorial type of the obtained polytope is independent of which vertices are cut. One can also easily see that for $k\geq 3$ this does not remain true.

Denote by $Q\equiv Q(n,k)$, V - its vertex set, and the symplicial polytope $Q^{'}$ is obtained from Q by only one suspension of a cone with a leading vertex v above its certain facet σ, which vertex set we will denote by $V(\sigma)$. Lets also consider $V^{'}$ - as a vertex set of $Q^{'}, W\subset V^{'}$ and K_W - a full subcomplex on the vertex set of W in the sense of [1, p. 44].

Due to the well known Hochster's formula (see. [1, Theorem 3.17]) lets try to ask the following question: what is happening with the reduced homology groups of the full subcomplexes K_W under some determined field of coefficients k when one goes from Q to $Q^{'}$? Therefore we are going to consider the only 4 logically possible cases (in what follows we have W as a proper subset of $V^{'}$)

(1) $v\in W, W\cap V(\sigma)\neq\emptyset$

If $V(\sigma)\subset W$, then $\|K^{'}_W\|\cong\|K_{W-\{v\}}\|$. If $W\cap V(\sigma)\neq V(\sigma)$, then we have:

$$K_{W-\{v\}}\cup K_{W\cap V(\sigma)\cup\{v\}}{}^{'}=K^{'}_W, K_{W-\{v\}}\cap K_{W\cap V(\sigma)\cup\{v\}}{}^{'}=K_{W\cap v(\sigma)}.$$

But $K_{W\cap V(\sigma)}$ and $K^{'}_{W\cap V(\sigma)\cup\{v\}}$ - in this case are, obviously, contractable. With the help of the Mayer-Vietoris exact sequence, we finally have:

$$\tilde{H}_i(K^{'}_W,k)\cong\tilde{H}_i(K_{W-\{v\}},k).$$

(2) $v\in W, W\cap V(\sigma)=\emptyset$

In this case it is easy to see that $K^{'}_W=K_{W-\{v\}}\sqcup\{v\}$. From this follows:

$$dim_k\tilde{H}_i(K^{'}_W,k)=\begin{cases} dim_k\tilde{H}_i(K_{W-\{v\}},k)+1, & \text{for } i=0;\\ dim_k\tilde{H}_i(K_{W-\{v\}},k), & \text{for } i>0.\end{cases}$$

(3) $v\notin W, V(\sigma)\subset W, W\neq V$

Consider a set Δ^σ of all subsets of $V(\sigma)$. Then as a boundary of this set we will naturally have $\delta=\Delta^\sigma-V(\sigma)$. It is clear that in this particular case we must obtain the following statements:

$$K^{'}_W\cup\Delta^\sigma=K_W, K^{'}_W\cap\Delta^\sigma=\delta,$$

(only the former facet F is vanishing.) But δ - is a symplicial $(n-2)-$sphere, Δ^{σ} - is a symplicial $(n-1)-$disk. Then with the help of the exact homology sequence with respect to $i=(n-1)$, one can soon obtain:

$$dim_k\tilde{H}_i(K_W^{'},k)=\begin{cases} dim_k\tilde{H}_i(K_W,k), & \text{for } i<(n-2);\\ dim_k\tilde{H}_i(K_W,k)+1, & \text{for } i=(n-2).\end{cases}$$

(4) $v\notin W, V(\sigma)\not\subset W$

In this case it is not hard to convince oneself, that $K_W^{'}\equiv K_W$. That's why we have:

$$dim_k\tilde{H}_i(K_W^{'},k)=dim_k\tilde{H}_i(K_W,k),\forall i.$$

Now we can formulate our main statement as follows:

Theorem 3.1. *For $n\geq 3$ the following formulae for bigraded Betti numbers associated with the simple polytopes $P(n,k)$ take place:*

$$\beta^{-i,2(i+1)}=i\cdot\binom{k+1}{i+1},$$
$$\beta^{-i,2(i+n-1)}=(k+1-i)\cdot\binom{k+1}{k+2-i},$$
$$\beta^{-i,2j}=0, j\neq i+1, i+n-1;$$

for $n=2$ the following takes place:

$$\beta^{-i,2(i+1)}=i\cdot\binom{k+1}{i+1}+(k+1-i)\cdot\binom{k+1}{k+2-i},$$
$$\beta^{-i,2j}=0, j\neq i+1.$$

Proof. The first formulae is proved in [4]; the second - its obvious consequence, if one notices the case of Poincare duality. The formulae above for the case of $n=2$ - are obtained from the analogous formulae for grater n with the help of dimension matters (for example, nonzero element of the Betti number table - is just a sum of corresponding 2 elements nonzero elements from the case of higher dimension). Lest prove the rest of the statement about the zero Betti numbers. More precise:

Theorem 3.2. *The following formula takes place:*

$$\tilde{H}_i(K_W,k)=0,\forall i\neq 0, n-2, \forall\emptyset\neq W\subsetneq V.$$

Proof. If $m=n+1$, then $P(n,k)$ - n-symplex, so K_W - is contactable. The induction statement: lets take that the conclusion of the theorem is true for all proper subsets of W of V for all $i\neq 0, n-2$, and consider W - as a proper subset of $V^{'}$. If $W=V^{'}-\{v\}=V$, then $K_W^{'}$ - is a symplicial $(n-1)-$disk and that makes the conclusion be true. If $W=\{v\}$, then we can say the same. In the common case, according to what is mentioned above, while considering the 4 cases, we obtain the following:

$$dim_k\tilde{H}_i(K_W^{'},k)=dim_k\tilde{H}_i(K_{W-\{v\}},k)=0;$$

□

□

From our observations it is easy to see, that the following takes place

Corollary 3.3. *Bigraded Betti numbers, associated with $P(n,k)$ and its dual one $Q(n,k)$, are independent of the combinatorial types of the latter ones, and the numbers, corresponding to $j=i+1$ are as well independent of the dimension of the total euclidean space.*

4. Final remarks

In the conclusion, lets consider the following interesting

Example 4.1. In [2, Theorem 6.3] the following result is stated:

Theorem 4.2. *Denote X - is a moment-angle-manifold, corresponding to $P(n,k)$. Then X is diffeomorphic to the connected sum of sphere products:*

$$\sharp_{j=1}^{k} j\cdot\binom{k+1}{j+1}S^{2+j}\times S^{2n+k-j-1}.$$

In accordance with our main statement, the coefficients in this formula give us an opportunity to find the full Betti numbers for X, for which computation the following formula could be used:

$$dim_k H^k(\mathcal{Z}_P)=\sum_{-i+2j=k}\beta^{-i,2j}(P).$$

From this fact we have:

$$b_3=\beta^{-1,4}=\binom{k+1}{2}=\binom{m-n}{2},$$

which coincides with the result of the main statement with $k=m-n-1, i=1, j=2$.

If we finally analyse the numerical examples, considered above, we can proclaim the following:

Conjecture 4.3. For the n-dimensional Stasheff polytope K_{n+2}, with $n\geq 2$ the following take place:

$$\beta^{-i,2(n+i-1)}=0, i=1,\cdots,2n-5,$$

$$\beta^{-(2n-4),2(3n-5)}=\begin{cases} n+3, & \text{if n - is even;}\\ \frac{n+3}{2}, & \text{if n - is odd.}\end{cases}$$

It is possible that "minimal" properties of such type can be observed also in the cases of other nestohedra (stellohedra, permutohedra, cyclohedra), as the author has information in the case of $n=3$, but he could not prove or even just formulate corresponding statements.

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